On Boundedness of Volumes and Birationality in Birational Geometry

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Preface

This is a Ph.D. thesis submitted to Graduate School of Mathematical Sciences, the University of Tokyo.

Throughout this thesis, we work over the field of complex numbers $\mathbb{C}$. We adopt the standard notations and definitions in [KaMaMa87] and [KoMo98], and will freely use them.

The aim of birational geometry is to classify all varieties up to birational equivalence. According to Minimal Model Program, minimal varieties and Fano varieties with mild singularities form fundamental classes in birational geometry. To understand these special classes of varieties, it is very natural and interesting to prove some boundedness results. The goal of this thesis is to collect my recent works in birational geometry centered around the theme of boundedness.

Chapter 1 contains a brief summary of the motivations, main problems, histories, and main results on boundedness of volumes and birationality.

Chapter 2 provides basic knowledge on volumes, Hirzebruch surfaces, non-klt centers, connectedness lemma, rational map defined by a Weil divisor, Reid's Riemann–Roch formula, and so on. Basic lemmas are also provided to support the following chapters.

Chapter 3 focuses on the boundedness of anti-canonical volumes. We prove Weak Borisov–Alexeev–Borisov Conjecture in dimension three which states that the anti-canonical volume of an ϵ-klt log Fano pair of dimension three is bounded from above. As a corollary, we give a different proof of boundedness of log Fano threefolds of fixed index.

Chapters 4 and 5 are devoted to the boundedness of birationality.

In Chapter 4, we investigate the pluri-anti-canonical linear systems of weak $\mathbb{Q}$-Fano 3-folds. We prove that, for a $\mathbb{Q}$-Fano 3-fold X, $|-mK_X|$ gives a birational map for $m \geq 39$, and for a weak $\mathbb{Q}$-Fano 3-fold X, $|-mK_X|$ gives a birational map for $m \geq 97$. We also consider the generic finiteness and prove that for a $\mathbb{Q}$-Fano 3-fold X, $|-mK_X|$ gives a generically finite map for $m \geq 28$. Plenty of examples are provided for discussing the optimality of these results.

In Chapter 5, we investigate minimal 3-fold X with numerically trivial canonical divisor and a nef and big Weil divisor L on X. We prove that $|mL|$ and $|K_X + mL|$ give birational maps for $m \geq 17$.

Chapters 3 and 5 are based on my preprints [Jiang14b, Jiang14a]. Chapter 4 is based on a joint work with Meng Chen [CJ14].

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Contents

1 Introduction

The aim of birational geometry is to classify all varieties up to birational equivalence. According to Minimal Model Program, minimal varieties and Fano varieties with mild singularities form fundamental classes in birational geometry. To understand these special classes of varieties, it is very natural and interesting to prove some boundedness results. In particular, we are interested in the boundedness of anti-log-canonical volumes of singular log Fano varieties and that of birationality of minimal 3-folds and $\mathbb{Q}$-Fano 3-folds.

1.1 Boundedness of anti-canonical volumes

Definition 1.1.1. A *pair* (X, Δ) consists of a normal projective variety X and an effective $\mathbb{Q}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. (X, Δ) is called a *log Fano pair* (resp. *weak log Fano pair*) if $-(K_X + \Delta)$ is ample (resp. nef and big). If $\dim X = 2$, we will use *del Pezzo* instead of Fano.

Definition 1.1.2. Let (X, Δ) be a pair. Let $f : Y \to X$ be a log resolution of (X, Δ), write

$$K_Y = f^*(K_X + \Delta) + \sum a_i F_i,$$

where F_i is a prime divisor. The coefficient a_i is called the *discrepancy* of F_i with respect to (X, Δ), and denoted by $a_{F_i}(X, \Delta)$. For some $\epsilon \in [0, 1]$, the pair (X, Δ) is called

(a) *ϵ-kawamata log terminal* (*ϵ-klt*, for short) if $a_i > -1 + \epsilon$ for all i;

(b) *ϵ-log canonical* (*ϵ-lc*, for short) if $a_i \geq -1 + \epsilon$ for all i;

(c) *terminal* if $a_i > 0$ for all f-exceptional divisors F_i.

Note that 0-klt (resp. 0-lc) is just klt (resp. lc) in the usual sense.

Definition 1.1.3. A variety X is *of ϵ-Fano type* if there exists an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is an ϵ-klt log Fano pair.

We are mainly interested in the boundedness of ϵ-Fano type varieties.

Definition 1.1.4. A collection of varieties $\{X_\lambda\}_{\lambda\in\Lambda}$ is said to be *bounded* if there exists $h : \mathcal{X} \to S$ a morphism of finite type of Neotherian schemes such that for each X_λ, $X_\lambda \simeq \mathcal{X}_s$ for some $s \in S$.

Our motivation is the following BAB Conjecture due to A. Borisov, L. Borisov, and V. Alexeev.

Conjecture 1.1.5 (BAB Conjecture)**.** *Fix $0 < \epsilon < 1$, an integer $n > 0$.*

Then the set of all n-dimensional ϵ-Fano type varieties is bounded.

BAB Conjecture is one of the most important conjecture in birational geometry and it is related to the termination of flips. As the approach to this conjecture, we are interested in the following much weak conjecture for anti-canonical volumes which is a consequence of BAB Conjecture.

Conjecture 1.1.6 (Weak BAB Conjecture)**.** *Fix $0 < \epsilon < 1$ and an integer $n > 0$.*

Then there exists a number $M(n, \epsilon)$ depending only on n and ϵ with the following property:

If (X, Δ) is an n-dimensional ϵ-klt log Fano pair, then

$$\mathrm{Vol}(-(K_X + \Delta)) = (-(K_X + \Delta))^n \leq M(n, \epsilon).$$

Further, if K_X is $\mathbb{Q}$-Cartier, then

$$\mathrm{Vol}(-K_X) \leq M(n, \epsilon).$$

BAB Conjecture was proved in dimension two by Alexeev [Ale94a] with a simplified argument by Alexeev–Mori [AM04]. In dimension three or higher, BAB Conjecture is still open. There are only some partial boundedness results. For example, we have boundedness of smooth Fano manifolds by Kollár–Miyaoka–Mori [KoMiMo92], that of terminal $\mathbb{Q}$-Fano $\mathbb{Q}$-factorial threefolds of Picard number one by Kawamata [Kaw92a], that of canonical $\mathbb{Q}$-Fano threefolds by Kollár–Miyaoka–Mori–Takagi [KMMT00], and that of toric varieties by Borisov–Borisov [BB92].

Weak BAB Conjecture in dimension two was treated by Alexeev [Ale94a], Alexeev–Mori [AM04], and Lai [Lai12]. Recently, the author [Jiang13] gave an optimal value for the number $M(2, \epsilon)$. For Weak BAB Conjecture in dimension three assuming that Picard number of X is one, an effective value of $M(3, \epsilon)$ was announced by Lai [Lai12]. For general case of dimension three and higher, Weak BAB Conjecture is still open.

As the main theorem in Chapter 3, we prove Weak BAB Conjecture in dimension three.

Theorem 1.1.7. *Weak BAB Conjecture holds for* $n = 3$*.*

As a consequence, we get a different proof of a result on the boundedness of log Fano varieties of fixed index in dimension three which was conjectured by Batyrev, and proved by A. Borisov [Bor01] in dimension three and Hacon–M^{c}Kernan–Xu [HMX14, Corollary 1.8] in arbitrary dimension.

Corollary 1.1.8. *Fix a positive integer* r*.*

Let $\mathcal{D}$ *be the set of all normal projective varieties* X*, where* $\dim X = 3$*,* K_X *is* $\mathbb{Q}$*-Cartier, and there exists an effective* $\mathbb{Q}$*-divisor* Δ *such that* (X, Δ) *is klt and* $-r(K_X + \Delta)$ *is Cartier and ample.*

Then $\mathcal{D}$ *forms a bounded family.*

1.1.1 Description of the proof

Firstly, we give an approach to Weak BAB Conjecture via Mori fiber spaces.

Definition 1.1.9. A projective morphism $X \to T$ between normal varieties is called a *Mori fiber space* if the following conditions hold:

(i) X is $\mathbb{Q}$-factorial with terminal singularities;

(ii) f is a *contraction*, i.e. $f_*\mathcal{O}_X = \mathcal{O}_T$;

(iii) $-K_X$ is ample over T;

(iv) $\rho(X/T) = 1$;

(v) $\dim X > \dim T$.

At this time, we say that X is with a *Mori fiber structure.*

We raise the following conjecture for Mori fiber spaces.

Conjecture 1.1.10 (Weak BAB Conjecture for Mori fiber spaces)**.** *Fix* $0 < \epsilon < 1$*, an integer* $n > 0$*.*

Then there exists a number $M(n, \epsilon)$ *depending only on* n *and* ϵ *with the following property:*

If X is an n-dimensional ϵ-Fano type variety with a Mori fiber structure, then

$$\mathrm{Vol}(-K_X) \leq M(n, \epsilon).$$

We prove the following theorem by using Minimal Model Program.

Theorem 1.1.11. *Weak BAB Conjecture holds for fixed ϵ and n if and only if Weak BAB Conjecture for Mori fiber spaces holds for fixed ϵ an n.*

By Theorem 1.1.11, to consider the boundedness of anti-canonical volumes of log Fano pairs, we only need to consider the ones with better singularities ($\mathbb{Q}$-factorial terminal singularities) and with additional structures (Mori fiber structures). This is the advantage of this theorem. In dimension two, this theorem appears as a crucial step to get the optimal value of $M(2, \epsilon)$ (c.f. [Jiang13]).

Restricting our interest to dimension three, we prove the following theorem.

Theorem 1.1.12. *Weak BAB Conjecture for Mori fiber spaces holds for $n = 3$.*

Theorem 1.1.7 follows from Theorems 1.1.11 and 1.1.12 directly.

To prove Theorem 1.1.12, we need to consider ϵ-Fano type 3-fold X with a Mori fiber structure $X \to T$. There are 3 cases:

(1) $\dim T = 0$, X is a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-folds with $\rho = 1$;

(2) $\dim T = 1$, $X \to T \simeq \mathbb{P}^1$ is a *del Pezzo fibration*, i.e. a general fiber is a smooth del Pezzo surface;

(3) $\dim T = 2$, $X \to T$ is a *conic bundle*, i.e. a general fiber is a smooth rational curve.

The second statement is implied by the following fact: if (X, Δ) is a klt log Fano pair, then X is rationally connected (see [Zha06, Theorem 1]), in particular, for any surjective morphism $X \to T$ to a normal curve, $T \simeq \mathbb{P}^1$.

In Case (1), X is bounded by Kawamata [Kaw92a], and the optimal bound of $\mathrm{Vol}(-K_X) = (-K_X)^3$ is 64 due to the classification on smooth Fano 3-folds of Iskovskikh and Mori–Mukai and by Namikawa's result [Nam97] (Gorenstein case) and Prokhorov [Pro07] (non-Gorenstein case).

We will mainly treat Cases (2) and (3).

One basic idea is to construct singular pairs which is not klt along fibers of $X \to T$. Then by Connectedness Lemma, we may find a non-klt center intersecting with the fibers. Finally by restricting on a general fiber, we get the bound after some arguments on lower dimensional varieties. But several difficulties arise here.

In Case (3), the difficulty arises in the construction of singular pair because we need to avoid components which are vertical over T. To do this, we need a good understanding of the singularities and boundedness of the surface T, which was done by several papers as [Ale94a], [MP08], and [Bir14].

In Case (2), the difficulty arises in the last step. After restricting on a general fiber, we need to bound the (generalized) log canonical thresholds on surfaces. So we are done by proving the following (generalized) Ambro's conjecture in dimension two.

Definition 1.1.13. Let (X, B) be a lc pair and $D \geq 0$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. The *log canonical threshold* of D with respect to (X, B) is

$$\mathrm{lct}(X, B; D) = \sup\{t \in \mathbb{Q} \mid (X, B + tD) \text{ is lc}\}.$$

For the use of this thesis, we need to consider the case when D is not effective. Let G be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor satisfying $G + B \geq 0$, The *generalized log canonical threshold* of G with respect to (X, B) is

$$\mathrm{glct}(X, B; G) = \sup\{t \in [0, 1] \cap \mathbb{Q} \mid (X, B + tG) \text{ is lc}\}.$$

Conjecture 1.1.14 (Ambro's conjecture)**.** *Fix $0 < \epsilon < 1$ and integer $n > 0$.*

Then there exists a number $\mu(n, \epsilon) > 0$ depending only on n and ϵ with the following property:

If (Y, B) is an ϵ-klt log Fano pair of dimension n, then

$$\inf\{\mathrm{lct}(Y, B; D) \mid D \sim_{\mathbb{Q}} -(K_Y + B), D \geq 0\} \geq \mu(n, \epsilon).$$

Note that we do not assume any special conditions on the coefficients of B. The left-hand side of the inequality is called *α-invariant* of (Y, B) which generalizes the concept of α-invariant of Tian for Fano manifolds in differential geometry (see [CMG14, CS08, Tian87]). Recently Ambro [Amb14] announced a proof of this conjecture assuming that (Y, B) is a toric pair where an explicit sharp number $\mu(n, \epsilon)$ was given. For the use of this paper, we need a stronger version of this conjecture where D may not be effective.

Conjecture 1.1.15 (generalized Ambro's conjecture)**.** *Fix $0 < \epsilon < 1$ and integer $n > 0$.*

Then there exists a number $\mu(n, \epsilon) > 0$ depending only on n and ϵ with the following property:

If (Y, B) is an ϵ-klt weak log Fano pair of dimension n and Y has at worst terminal singularities, then

$$\inf\{\mathrm{glct}(Y, B; G) \mid G \sim_{\mathbb{Q}} -(K_Y + B), G + B \geq 0\} \geq \mu(n, \epsilon).$$

Note that Conjecture 1.1.14 follows from Conjecture 1.1.15 easily after taking terminalization of (Y, B).

We prove the conjecture in dimension two by following some ideas in the proof of BAB Conjecture in dimension two ([Ale94a, AM04]). But it seems that this conjecture does not follow from BAB Conjecture trivially.

Theorem 1.1.16. *Conjecture 1.1.15 holds for $n = 2$.*

For the proof of Corollary 1.1.8, we basically follow the idea in [Bor01] to bound the Hilbert polynomials by [KoMa83].

1.2 Boundedness of birationality

Definition 1.2.1. A normal projective variety X is called a *weak $\mathbb{Q}$-Fano variety* if X has at worst $\mathbb{Q}$-factorial terminal singularities and the anti-canonical divisor $-K_X$ is nef and big. A weak $\mathbb{Q}$-Fano variety is said to be *$\mathbb{Q}$-Fano* if $-K_X$ is $\mathbb{Q}$-ample and the Picard number $\rho(X) = 1$.

Definition 1.2.2. A normal projective variety X is said to be *minimal* if X has at worst $\mathbb{Q}$-factorial terminal singularities and the canonical divisor K_X is nef.

According to Minimal Model Program, $\mathbb{Q}$-Fano varieties and minimal varieties form fundamental classes in birational geometry.

Given an n-dimensional normal projective variety X with mild singularities and a big Weil divisor L on X, we are interested in the geometry of the rational map $\Phi_{|mL|}$ defined by the linear system $|mL|$. By definition, $\Phi_{|mL|}$ is birational onto its image when m is sufficiently large. Therefore it is interesting to find such a practical number $m(n)$, depending only on $\dim X$, which stably guarantees the birationality of $\Phi_{|mL|}$. In fact, the following three special cases are the most interesting:

(i) K_X is nef and big, $L = K_X$;

(ii) $K_X \equiv 0$, L is an arbitrary nef and big Weil divisor;

(iii) $-K_X$ is nef and big, $L = -K_X$.

It is an interesting exercise to deal the case X being a smooth curve or surface.

Theorem 1.2.3 (c.f. Bombieri [Bom73], Reider [Reider88])**.** *Let S be a smooth surface.*

(i) If K_S is nef and big, then $|mK_S|$ gives a birational map for $m \geq 5$;

(ii) If $K_S \equiv 0$, then $|mL|$ gives a birational map for $m \geq 3$ and L an arbitrary nef and big divisor;

(iii) If $-K_S$ is nef and big, then $|-mK_S|$ gives a birational map for $m \geq 3$.

For a 3-fold X, when X is smooth, these cases were treated by Matsuki [Mat86], Ando [Ando87], Fukuda [Fuk91], Oguiso [Ogu91], and many others, and we have the following known results.

Theorem 1.2.4 (Matsuki [Mat86], Fukuda [Fuk91])**.** *Let X be a smooth 3-fold.*

(i) If K_X is nef and big, then $|mK_X|$ gives a birational map for $m \geq 6$;

(ii) If $K_X \equiv 0$, then $|mL|$ gives a birational map for $m \geq 6$ and L an arbitrary nef and big divisor;

(iii) If $-K_X$ is nef and big, then $|-mK_X|$ gives a birational map for $m \geq 4$.

When X is a 3-fold with $\mathbb{Q}$-factorial terminal singularities, Case (i) was systematically treated by J. A. Chen and M. Chen [CC10a, CC10b, CC13].

Theorem 1.2.5 (Chen–Chen [CC13])**.** *Let X be a minimal 3-fold of general type (i.e K_X is nef and big), then $|mK_X|$ gives a birational map for $m \geq 61$.*

We are going to treat Cases (ii) and (iii) systematically.

1.2.1 $\mathbb{Q}$-Fano threefolds

In Chapter 4, for a weak $\mathbb{Q}$-Fano 3-fold X, the *anti-m-canonical map* φ_{-m} is the rational map defined by the linear system $|-mK_X|$. Such a number m_3 that stably guarantees the birationality of φ_{-m_3} exists due to the boundedness of $\mathbb{Q}$-Fano 3-folds, which was proved by Kawamata [Kaw92a], and the boundedness of weak $\mathbb{Q}$-Fano 3-folds proved by Kollár–Miyaoka–Mori–Takagi [KMMT00]. It is natural to consider the following problem.

Problem 1.2.6. *Find the optimal constant c such that φ_{-m} is birational onto its image for all $m \geq c$ and for all (weak) $\mathbb{Q}$-Fano 3-folds.*

The following example tells us that $c \geq 33$.

Example 1.2.7 ([IF00, List 16.6, No.95])**.** The general weighted hypersurface $X_{33} \subset \mathbb{P}(1, 5, 6, 22, 33)$ is a $\mathbb{Q}$-Fano 3-fold. It is clear that φ_{-m} is birational onto its image for $m \geq 33$, but φ_{-32} fails to be birational.

It is worthwhile to compare the birational geometry induced from $|mK|$ on varieties of general type with the geometry induced from $|-mK|$ on (weak) $\mathbb{Q}$-Fano varieties. An obvious feature on Fano varieties is that the behavior of φ_{-m} is not necessarily birationally invariant. For example, consider degree 2 (rational) del Pezzo surface S_2 and $\mathbb{P}^2$, $|-K_{\mathbb{P}^2}|$ gives a birational map but $|-K_{S_2}|$ does not. This causes difficulties in studying Problem 1.2.6. In fact, even if in dimension 3, there is no known practical upper bound for c in written records.

When X is smooth, we may take $c = 4$ according to Ando [Ando87] and Fukuda [Fuk91]. When X has terminal singularities, Problem 1.2.6 was treated by M. Chen in [Chen11], where an effective upper bound of c in terms of the Gorenstein index of X is proved (cf. [Chen11, Theorem 1.1]). Since, however, the Gorenstein index of a weak $\mathbb{Q}$-Fano 3-fold can be as large as "840" (see Proposition 4.1.1), the number "$3 \times 840 + 10 = 2530$" obtained in [Chen11, Theorem 1.1] is far from being optimal. It turns out that Problem 1.2.6 is closely related to the following problem (cf. [Chen11, Theorem 4.5]).

Problem 1.2.8. *Given a (weak) $\mathbb{Q}$-Fano 3-fold X, can one find the least positive integer $\delta_1 = \delta_1(X)$ such that $\dim \overline{\varphi_{-\delta_1}(X)} > 1$?*

Problem 1.2.8 is parallel to the following question on 3-folds of general type:

> *Let Y be a 3-fold of general type on which $|nK_Y|$ is composed with a pencil of surfaces for some fixed integer $n > 0$. Can one find an integer m (bounded from above by a function in terms of n) so that $|mK_Y|$ is not composed with a pencil any more?*

This question was solved by Kollár [Kol86] who proved that one may take $m \leq 11n + 5$. The result is a direct application of the semi-positivity of $f_*\omega^l_{Y/B}$ since, modulo birational equivalence, one may assume that there is a fibration $f : Y \longrightarrow B$ onto a curve B. As far as we know, there is still no known analogy of Kollár's method in treating $\mathbb{Q}$-Fano varieties.

Firstly, we shall prove the following theorem.

Theorem 1.2.9. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then there exists an integer $n_1 \leq 10$ such that $\dim \overline{\varphi_{-n_1}(X)} > 1$.*

Theorem 1.2.9 is close to be optimal due to the following example.

Example 1.2.10 ([IF00, List 16.7, No.85])**.** Consider the general codimension 2 weighted complete intersection $X := X_{24,30} \subset \mathbb{P}(1, 8, 9, 10, 12, 15)$ which is a $\mathbb{Q}$-Fano 3-fold. Then $\dim \overline{\varphi_{-9}(X)} > 1$ while $\dim \overline{\varphi_{-8}(X)} = 1$ since $h^0(-8K_X) = 2$.

In fact, theoretically, there are only 4 possible weighted baskets for which we need to take $n_1 = 10$ (see Remark 4.2.13 and Subsection 4.2.6 for more details and discussions). Theorem 1.2.9 allows us to prove the following result.

Theorem 1.2.11. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then φ_{-m} is birational onto its image for all $m \geq 39$.*

In particular, as a by-product we have the following corollary which is optimal.

Corollary 1.2.12. *Let X be a $\mathbb{Q}$-Fano 3-fold.*

(i) If $h^0(-K_X) \geq 3$, then φ_{-m} is birational onto its image for all $m \geq 6$;

(ii) If $h^0(-K_X) = 2$, then φ_{-m} is birational onto its image for all $m \geq 21$.

The optimality is shown by the general weighted hypersurfaces $X_{12} \subset \mathbb{P}(1,1,1,4,6)$ and $X_{42} \subset \mathbb{P}(1,1,6,14,21)$ ([IF00, List 16.6, No.14, No.88]).

T. Sano suggested that we can consider the generic finiteness of φ_{-m}, and we get the following result.

Theorem 1.2.13. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then φ_{-m} is generically finite onto its image for all $m \geq 28$.*

Note that in Example 1.2.7, φ_{-22} is generically finite onto its image but φ_{-21} is not.

A key point in proving Theorem 1.2.9 is that we have $\rho(X) = 1$, which is not the case for arbitrary weak $\mathbb{Q}$-Fano 3-folds. Therefore we should study weak $\mathbb{Q}$-Fano 3-folds in an alternative way. Our result is as follows.

Theorem 1.2.14. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Then $\dim \overline{\varphi_{-n_2}(X)} > 1$ for all $n_2 \geq 71$.*

Theorem 1.2.14 allows us to study the birationality.

Theorem 1.2.15. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Then φ_{-m} is birational onto its image for all $m \geq 97$.*

Also we can prove similar result on generic finiteness, but it seems not so interesting.

1.2.2 Minimal 3-folds with $K \equiv 0$

In Chapter 5, for a minimal 3-fold X with $K_X \equiv 0$ and an arbitrary nef and big Weil divisor L on X, we are interested in the rational map $\Phi_{|mL|}$ defined by the linear system $|mL|$. If X is smooth, then $|mL|$ gives a birational map for $m \geq 6$ by Fukuda [Fuk91]. If X is with Gorenstein terminal singularities and $q(X) := h^1(\mathcal{O}_X) = 0$, then $|mL|$ gives a birational map for $m \geq 5$ by Oguiso–Peternell [OP95].

The motivation of Chapter 5 is to systematically study the birational geometry of minimal 3-fold with $K \equiv 0$. For an arbitrary nef and big Weil divisor L on X, we investigate the birationality of the linear system $|mL|$. For special interest, we also investigate the birationality of the adjoint linear system $|K_X + mL|$.

The difficulty arises from the singularities of X, and the assumption that L is only a Weil divisor. If we assume that L is Cartier, then the problem becomes relatively easy and can be treated by the method of Fukuda [Fuk91] using Reider's theorem [Reider88]. On the other hand, fortunately, the singularities of minimal 3-folds with $K \equiv 0$ is not so complicated due to Kawamata [Kaw86] and Morrison [Mor86], and this makes it possible to deal with the birationality problem.

We prove the following theorem.

Theorem 1.2.16. *Let X be a minimal 3-fold with $K_X \equiv 0$ and a nef and big Weil divisor L. Then $|mL|$ and $|K_X + mL|$ give birational maps for all $m \geq 17$.*

In fact, we prove a more general theorem.

Theorem 1.2.17. *Let X be a minimal 3-fold with $K_X \equiv 0$, a nef and big Weil divisor L, and a Weil divisor $T \equiv 0$. Then $|K_X + mL + T|$ gives a birational map for all $m \geq 17$.*

Moreover, by Log Minimal Model Program, the assumption that L is nef can be weaken. We say that a divisor D *has no stable base components* if $|mD|$ has no base components for sufficiently divisible m.

Theorem 1.2.18. *Let X be a minimal 3-fold with $K_X \equiv 0$, a big Weil divisor L without stable base components, and a Weil divisor $T \equiv 0$. Then $|K_X + mL + T|$ gives a birational map for all $m \geq 17$. In particular, $|mL|$ and $|K_X + mL|$ give birational maps for all $m \geq 17$.*

As a by-product, we prove a direct generalization of Fukuda [Fuk91] and Oguiso–Peternell [OP95] which is optimal by the general weighted hypersurface $X_{10} \subset \mathbb{P}(1, 1, 1, 2, 5)$.

Theorem 1.2.19 (=Theorem 5.2.2)**.** *Let X be a minimal Gorenstein 3-fold with $K_X \equiv 0$, a nef and big Weil divisor L, and a Weil divisor $T \equiv 0$. Then $|K_X + mL + T|$ gives a birational map for all $m \geq 5$.*

2
Preliminaries

2.1 Volumes

Definition 2.1.1. Let X be an n-dimensional projective variety and D be a Cartier divisor on X. The *volume* of D is the real number

$$\mathrm{Vol}(D) = \limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

Note that the limsup is actually a limit. Moreover by the homogenous property of the volume, we can extend the definition to $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. Note that if D is a nef $\mathbb{Q}$-divisor, then $\mathrm{Vol}(D) = D^n$.

For more background on volumes, see [Laz04, 11.4.A].

2.2 Hirzebruch surfaces

We recall some basic properties of the Hirzebruch surfaces $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$. Denote by h (resp. f) the class in Pic $\mathbb{F}_n$ of the tautological bundle $\mathcal{O}_{\mathbb{F}_n}(1)$ (resp. of a fiber). Then Pic $\mathbb{F}_n = \mathbb{Z}h \oplus \mathbb{Z}f$ with $f^2 = 0$, $f \cdot h = 1$, $h^2 = n$. If $n > 0$, there is a unique irreducible curve $\sigma_n \subset \mathbb{F}_n$ such that $\sigma_n \sim h - nf$, $\sigma_n^2 = -n$. For $n = 0$, we can also choose one curve whose class in Pic $\mathbb{F}_0$ is h and denote it by σ_0. Note that

$$-K_{\mathbb{F}_n} \sim 2h - (n-2)f \sim 2\sigma_n + (n+2)f.$$

Lemma 2.2.1. *For an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_{\mathbb{F}_n}$ and a fiber f,* $\mathrm{mult}_f D \leq n + 2$.

Proof. Since $D - (\text{mult}_f D)f$ is effective, $(D - (\text{mult}_f D)f) \cdot h \geq 0$. On the other hand, $(D - (\text{mult}_f D)f) \cdot h = n + 2 - \text{mult}_f D$. $\square$

Lemma 2.2.2. *Let $T = \mathbb{P}^2$ or $\mathbb{F}_n$, then for an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_T$ and a point Q, $\text{mult}_Q D \leq n + 4$ holds. Moreover, if we write $D = \sum_j b_j D_j$ by its components and assume that $b_j \leq 1$ for all j, then $\sum_j b_j \leq 4$.*

Proof. If $T = \mathbb{P}^2$, taking a general line L through Q, we have

$$3 = (D \cdot L) \geq \text{mult}_Q(D).$$

If $T = \mathbb{F}_n$, take f be the fiber passing through Q, by Lemma 2.2.1 and intersection theory, we have

$$2 = D \cdot f \geq \text{mult}_Q D - \text{mult}_f D \geq \text{mult}_Q D - n - 2.$$

For the latter statement, if $T = \mathbb{F}_n$, then the conclusion follows by [AM04, Lemma 1.4]. If $T = \mathbb{P}^2$, then $\sum b_j \leq 3$ by degree computation. $\square$

2.3 Non-klt centers and connectedness lemma

Definition 2.3.1. Let X be a normal projective variety and Δ be a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log resolution of (X, Δ), write

$$K_Y = f^*(K_X + \Delta) + \sum a_i F_i,$$

where F_i is a prime divisor. F_i is called a *non-klt place* if $a_i \leq -1$. A subvariety $V \subset X$ is called a *non-klt center* of (X, Δ) if it is the image of a non-klt place. The *non-klt locus* $\text{Nklt}(X, \Delta)$ is the union of all non-klt centers of (X, Δ). A non-klt center is *maximal* if it is an irreducible component of $\text{Nklt}(X, \Delta)$.

The following lemma suggests a standard way to construct non-klt centers.

Lemma 2.3.2 (cf. [KoMo98, Lemma 2.29])**.** *Let (X, Δ) be a pair and $Z \subset X$ be a close subvariety of codimesion k such that Z is not contained in the singular locus of X. If $\text{mult}_Z \Delta \geq k$, then Z is a non-klt center of (X, Δ).*

Recall that the *multiplicity* $\text{mult}_Z F$ of a divisor F along a subvariety Z is defined by the multiplicity $\text{mult}_x F$ of F at a general point $x \in Z$.

Unfortunately, the converse of Lemma 2.3.2 is not true unless $k = 1$. Usually we do not have good estimations for the multiplicity along a non-klt center but the following lemma.

Lemma 2.3.3 (cf. [Laz04, Theorem 9.5.13])**.** *Let (X, Δ) be a pair and $Z \subset X$ be a non-klt center of (X, Δ) such that Z is not contained in the singular locus of X. Then* $\mathrm{mult}_Z \Delta \geq 1$.

If we assume some simple normal crossing condition on the boundary, we can get more information on the multiplicity along a non-klt center. For simplicity, we just consider surfaces.

Lemma 2.3.4 (cf. [McK02, 4.1 Lemma])**.** *Fix $0 < e < 1$. Let S be a smooth surface, B be an effective $\mathbb{Q}$-divisor, and D be a (not necessarily effective) simple normal crossing supported $\mathbb{Q}$-divisor. Assume that coefficients of D are at most e and $\mathrm{mult}_P B \leq 1 - e$ for some point P, then for arbitrary divisor E centered on P over S, $a_E(S, B + D) \geq -e$. In particular, if Z is a non-klt center of $(S, B + D)$ and coefficients of D are at most e, then* $\mathrm{mult}_Z B > 1 - e$.

Proof. By taking a sequence of point blow-ups, we can get the divisor E. Consider the blow-up at P, we have $f : S_1 \to S$ with $K_{S_1} + B_1 + D_1 + mE_1 = f^*(K_S + B + D)$ where B_1 and D_1 are the strict transforms of B and D respectively, and E_1 is the exceptional divisor with $m = \mathrm{mult}_P(B + D) - 1 \leq 1 - e + 2e - 1 = e$. Now $D_1 + mE_1$ is again simple normal crossing supported and $\mathrm{mult}_Q B_1 \leq \mathrm{mult}_P B$ for $Q \in E_1$. Hence by induction on the number of blow-ups, we conclude that the coefficient of E is at most e and hence $a_E(S, B + D) \geq -e$. □

We have the following connectedness lemma of Kollár and Shokurov for non-klt locus (cf. Shokurov [Sho93], Kollár [Kol+92, 17.4]).

Theorem 2.3.5 (Connectedness Lemma)**.** *Let $f : X \to Z$ be a proper morphism of normal varieties with connected fibers and D is a $\mathbb{Q}$-divisor such that $-(K_X + D)$ is $\mathbb{Q}$-Cartier, f-nef and f-big. Write $D = D^+ - D^-$ where D^+ and D^- are effective with no common components. If D^- is f-exceptional (i.e. all of its components have image of codimension at least 2), then* $\mathrm{Nklt}(X, D) \cap f^{-1}(z)$ *is connected for any $z \in Z$.*

Remark 2.3.6. There are two main cases of interest of Connectedness Lemma:

(i) Z is a point and (X, D) is a weak log Fano pair. Then $\mathrm{Nklt}(X, D)$ is connected.

(ii) $f : X \to Z$ is birational, (Z, B) is a log pair and $K_X + D = f^*(K_Z + B)$.

2.4 Rational map defined by a Weil divisor

For two linear systems $|A|$ and $|B|$, we write $|A| \preceq |B|$ if

$$|B| \supset |A| + \text{fixed effective divisor}.$$

In particular, if $A \leq B$ as divisors, then $|A| \preceq |B|$.

Consider an integral $\mathbb{Q}$-Cartier Weil divisor D on X with $h^0(X, D) \geq 2$. We study the rational map defined by $|D|$, say

$$X \overset{\Phi_D}{\dashrightarrow} \mathbb{P}^{h^0(D)-1}$$

which is not necessarily well-defined everywhere. By Hironaka's big theorem, we can take successive blow-ups $\pi : Y \to X$ such that:

(i) Y is smooth projective;

(ii) the movable part $|M|$ of the linear system $|\lfloor \pi^*(D) \rfloor|$ is base point free and, consequently, the rational map $\gamma := \Phi_D \circ \pi$ is a morphism;

(iii) the support of the union of $\pi_*^{-1}(D)$ and the exceptional divisors of π is of simple normal crossings.

Let $Y \xrightarrow{f} \Gamma \xrightarrow{s} Z$ be the Stein factorization of γ with $Z := \gamma(Y) \subset \mathbb{P}^{h^0(D)-1}$. We have the following commutative diagram.

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, "\gamma"] & \Gamma \arrow[d, "s"] \\
X \arrow[r, dashed, "\Phi_D"] & Z
\end{tikzcd}$$

Case (f_{np}). If $\dim(\Gamma) \geq 2$, a general member S of $|M|$ is a smooth projective surface by Bertini's theorem. We say that $|D|$ *is not composed with a pencil of surfaces.*

Case (f_{p}). If $\dim(\Gamma) = 1$, i.e. $\dim \overline{\Phi_D(X)} = 1$, a general fiber S of f is an irreducible smooth projective surface by Bertini's theorem. We may write

$$M = \sum_{i=1}^{a} S_i \equiv aS$$

where S_i is a smooth fiber of f for all i. We say that $|D|$ *is composed with a pencil of surfaces.* It is clear that $a \geq h^0(D) - 1$. Furthermore,

$a = h^0(D) - 1$ if and only if $\Gamma \cong \mathbb{P}^1$, and then we say that $|D|$ *is composed with a rational pencil of surfaces*. In particular, if $q(X) = 0$, then $\Gamma \cong \mathbb{P}^1$ since $g(\Gamma) \leq q(Y) = q(X) = 0$. We can write

$$|D| = |nS'| + E,$$

where $|S'| = |\pi_* S|$ is an irreducible rational pencil, $|nS'|$ is the movable part, and E is the fixed part. And we collect a couple of basic facts about rational pencils as follows.

Lemma 2.4.1. *Keep the same notation as above. If $|D| = |nS'| + E$ is composed with a rational pencil of surfaces, then $n = h^0(D) - 1$.*

Lemma 2.4.2. *If $|D_1| = |k_1 S_1| + E_1$ and $|D_2| = |k_2 S_2| + E_2$ are composed with rational pencils of surfaces and $D_1 \leq D_2$, then $|S_1| = |S_2|$.*

Proof. Since $D_1 \leq D_2$, we have $\mathrm{Mov}|D_1| \leq \mathrm{Mov}|D_2|$. Hence $|S_1| \preceq |k_2 S_2|$. Thus $|S_1| \preceq |S_2|$ by the irreducibility of $|S_1|$. Then by $h^0(S_1) = h^0(S_2) = 2$ and $|S_1|, |S_2|$ are movable, we have $|S_1| = |S_2|$. $\square$

We say that $|D|$ and $|D'|$ are *composed with the same pencil* if $|D|$ and $|D'|$ are composed with pencils and they define the same fibration structure $Y \to \Gamma$.

Define

$$\iota = \iota(D) := \begin{cases} 1, & \text{Case } (f_{\mathrm{np}}); \\ a, & \text{Case } (f_{\mathrm{p}}). \end{cases}$$

Clearly, in both cases, $M \equiv \iota S$ with $\iota \geq 1$.

Definition 2.4.3. For both Case (f_{np}) and Case (f_{p}), we call S *a generic irreducible element of* $|M|$.

We may also define "a generic irreducible element" of a moving linear system on any surface in the similar way.

2.5 Reid's Riemann–Roch formula

Let X be a 3-fold with at most $\mathbb{Q}$-factorial terminal singularities. Denote by r_X or $i(X)$ the *Gorenstein index* or *local index* of X, i.e. the Cartier index of K_X. By Kawamata [Kaw88, Corollary 5.2], for arbitrary Weil divisor D on X, $i(X)D$ is a Cartier divisor.

A *basket* B is a collection of pairs of integers (permitting weights), say $\{(b_i, r_i) \mid i = 1, \cdots, s; b_i \text{ is coprime to } r_i\}$. For simplicity, we will alternatively write a basket as follows, say

$$B = \{(1,2),(1,2),(2,5)\} = \{2 \times (1,2),(2,5)\}.$$

Let X be a 3-fold with $\mathbb{Q}$-factorial terminal singularities. According to Reid [Reid87], for a Weil divisor D on X,

$$\chi(D) = \chi(\mathcal{O}_X) + \frac{1}{12}D(D-K_X)(2D-K_X) + \frac{1}{12}(D \cdot c_2) + \sum_Q c_Q(D),$$

where the last sum runs over Reid's basket of orbifold points. If the orbifold point Q is of type $\frac{1}{r_Q}(1,-1,b_Q)$ and $i_Q = i_Q(D)$ is the local index of divisor D at Q (i.e. $D \sim i_Q K_X$ around Q, $0 \le i_Q < r$), then

$$c_Q(D) = -\frac{i_Q(r_Q^2-1)}{12r_Q} + \sum_{j=0}^{i_Q-1} \frac{\overline{jb_Q}(r_Q - \overline{jb_Q})}{2r_Q}.$$

Here the symbol $\bar{\cdot}$ means the smallest residue mod r_Q and $\sum_{j=0}^{-1} := 0$. We can write Reid's basket as $B_X = \{(b_Q, r_Q)\}_Q$. Note that we may assume $0 < b_Q \le \frac{r_Q}{2}$. And recall that $r_X = i(X) = \text{l.c.m.}\{r_Q \in B_X\}$.

Write

$$\chi_{\text{sing}}(D) := \sum_Q c_Q(D) \text{ and}$$

$$\chi_{\text{reg}}(D) := \chi(\mathcal{O}_X) + \frac{1}{12}D(D-K_X)(2D-K_X) + \frac{1}{12}(D \cdot c_2).$$

2.5.1 Weak $\mathbb{Q}$-Fano 3-folds

Let X be a weak $\mathbb{Q}$-Fano 3-fold. For any positive integer m, the number $P_{-m}(X) := h^0(X, \mathcal{O}_X(-mK_X))$ is called the *m-th anti-plurigenus* of X. Clearly, since $-K_X$ is nef and big, Kawamata–Viehweg vanishing theorem [KaMaMa87, Theorem 1-2-5] implies

$$h^i(-mK_X) = h^i(X, K_X - (m+1)K_X) = 0$$

for all $i > 0$ and $m \ge 0$. In particular, $\chi(\mathcal{O}_X) = 1$.

We make some remarks here on how to compute the term $c_Q(D)$:

(1) If $D = nK_X$ for $n \in \mathbb{Z}$, we take $i = \overline{n}$ (mod r) and then

$$c_Q(nK_X) = c_Q(iK_X) = -\frac{i(r^2-1)}{12r} + \sum_{j=0}^{i-1} \frac{\overline{jb}(r-\overline{jb})}{2r}.$$

(2) If $D = tK_X$ for $t \in \mathbb{Z}^+$, then it is easy to see that

$$c_Q(tK_X) = -\frac{t(r^2-1)}{12r} + \sum_{j=0}^{t-1} \frac{\overline{jb}(r-\overline{jb})}{2r}.$$

(3) By Reid's formula, Kawamata–Veihweg vanishing theorem and Serre duality, we have, for any $n > 0$,

$$\begin{aligned} P_{-n}(X) = & -\chi(\mathcal{O}_X((n+1)K_X)) \\ = & \frac{1}{12}n(n+1)(2n+1)(-K_X^3) + (2n+1) - l(-n) \end{aligned}$$

where $l(-n) = l(n+1) = \sum_i \sum_{j=1}^n \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i}$ and the sum runs over Reid's basket of orbifold points

$$B_X = \{(b_i, r_i) \mid i = 1, \cdots, s; 0 < b_i \leq \frac{r_i}{2}; b_i \text{ is coprime to } r_i\}.$$

The above formula can be rewritten as:

$$\begin{aligned} P_{-1} &= \frac{1}{2}(-K_X^3 + \sum_i \frac{b_i^2}{r_i}) - \frac{1}{2}\sum_i b_i + 3, \\ P_{-m} - P_{-(m-1)} &= \frac{m^2}{2}(-K_X^3 + \sum_i \frac{b_i^2}{r_i}) - \frac{m}{2}\sum_i b_i + 2 - \Delta^m \end{aligned}$$

where $\Delta^m = \sum_i (\frac{\overline{b_i m}(r_i - \overline{b_i m})}{2r_i} - \frac{b_i m(r_i - b_i m)}{2r_i})$ for any $m \geq 2$.

2.5.2 Minimal 3-folds with $K \equiv 0$

Let X be a minimal 3-fold with $K_X \equiv 0$. Note that for arbitrary nef and big Weil divisor H, Kawamata–Viehweg vanishing theorem [KaMaMa87, Theorem 1-2-5] implies

$$h^i(H) = h^i(K_X + (H - K_X)) = 0$$

for all $i > 0$. For a nef and big Weil divisor L and a Weil divisor $T \equiv 0$, Reid's formula gives

$$h^0(mL+T) = \chi(\mathcal{O}_X) + \frac{m^3}{6}L^3 + \frac{m}{12}(L \cdot c_2) + \sum_Q c_Q(mL+T).$$

We make some remarks on estimating this formula. Recall that by Miyaoka [Miy87], c_2 is pseudo-effective and hence $(L \cdot c_2) \geq 0$ holds. Also Reid's formula or Kawamata [Kaw86, Theorem 2.4] gives

$$\chi(\mathcal{O}_X) = \sum_Q \frac{r_Q^2 - 1}{24 r_Q}. \tag{2.5.1}$$

We define

$$\lambda(L) := \frac{1}{6}L^3 + \frac{1}{12}(L \cdot c_2).$$

Note that $\lambda(L)$ is a numerical invariant of L. We can rewrite Reid's formula as following:

$$h^0(mL+T) = \chi(\mathcal{O}_X) + \frac{m^3 - m}{6}L^3 + m\lambda(L) + \sum_Q c_Q(mL+T).$$

And we have the following lemma.

Lemma 2.5.1. *$i(X)\lambda(L) \in \mathbb{Z}_{>0}$. In particular, $\lambda(L) \geq \frac{1}{i(X)}$.*

Proof. For a singular point Q of type (b, r), note that the local index of $L + iK_X$ at Q runs over $\{0, 1, \cdots, r-1\}$ if so does i. Hence we have

$$\begin{aligned}
&\sum_{i=0}^{r-1} c_Q(L + iK_X) \\
= &\sum_{i=0}^{r-1} \Big(-\frac{i(r^2-1)}{12r} + \sum_{j=0}^{i-1} \frac{\overline{jb}(r - \overline{jb})}{2r} \Big) \\
= &-\frac{(r-1)(r^2-1)}{24} + \sum_{i=1}^{r-1}\sum_{j=0}^{i-1} \frac{\overline{jb}(r-\overline{jb})}{2r} \\
= &-\frac{(r-1)(r^2-1)}{24} + \sum_{j=0}^{r-2}\sum_{i=j+1}^{r-1} \frac{\overline{jb}(r-\overline{jb})}{2r} \\
= &-\frac{(r-1)(r^2-1)}{24} + \sum_{j=1}^{r-2} (r-1-j)\frac{\overline{jb}(r-\overline{jb})}{2r}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{(r-1)(r^2-1)}{24} + \sum_{k=2}^{r-1}(k-1)\frac{\overline{kb}(r-\overline{kb})}{2r} \qquad\qquad\qquad\qquad (k = r-j)\\
&= -\frac{(r-1)(r^2-1)}{24} + \frac{1}{2}\sum_{k=1}^{r-1}((r-1-k)+(k-1))\frac{\overline{kb}(r-\overline{kb})}{2r}\\
&= -\frac{(r-1)(r^2-1)}{24} + \frac{r-2}{2}\sum_{k=0}^{r-1}\frac{\overline{kb}(r-\overline{kb})}{2r}\\
&= -\frac{(r-1)(r^2-1)}{24} + \frac{r-2}{2}\sum_{j=0}^{r-1}\frac{j(r-j)}{2r}\\
&= -\frac{r^2-1}{24}.
\end{aligned}
$$

Hence by Reid's formula,

$$
\begin{aligned}
&\sum_{i=0}^{i(X)-1} h^0(L+iK_X)\\
&= \sum_{i=0}^{i(X)-1}\Big(\chi(\mathcal{O}_X)+\lambda(L)+\sum_Q c_Q(L+iK_X)\Big)\\
&= i(X)\chi(\mathcal{O}_X)+i(X)\lambda(L)+\sum_Q\Big(-\frac{r_Q^2-1}{24}\cdot\frac{i(X)}{r_Q}\Big)\\
&= i(X)\lambda(L).
\end{aligned}
$$

Hence $i(X)\lambda(L)\in\mathbb{Z}$. On the other hand, $\lambda(L)>0$ since L is nef and big. $\square$

2.6 Intersection numbers and a non-pencil criterion

We have the following lemma for intersection numbers.

Lemma 2.6.1. *Let X be a normal projective 3-fold with $\mathbb{Q}$-factorial terminal singularities. Recall that $i(X)$ is the local index of X, i.e. the Cartier index of K_X. Then for Weil divisors D_1, D_2, and D_3 on X, $(i(X)D_1\cdot D_2\cdot D_3)\in\mathbb{Z}$. In particular, if L is a nef and big Weil divisor on X, then $L^3\geq\frac{1}{i(X)}$.*

Proof. Recall that by Kawamata [Kaw88, Corollary 5.2], $i(X)D_1$ is Cartier. Take a resolution of isolated singularities $\phi: W\to X$. We may write $K_W=$

$\phi^*(K_X)+\Delta$ where Δ is an exceptional effective $\mathbb{Q}$-divisor over those isolated terminal singularities on X. Denote by D_i' the strict transform of D_i on W for $i=1,2,3$. By intersection theory, we have

$$\begin{aligned}&(i(X)D_1\cdot D_2\cdot D_3)_X\\ =\ &(\phi^*(i(X)D_1)\cdot\phi^*(D_2)\cdot D_3')_W\\ =\ &(\phi^*(i(X)D_1)\cdot D_2'\cdot D_3')_W\end{aligned}$$

is an integer. □

As a corollary, we give a criterion for a linear system not composing with a pencil of surfaces by looking at its Hilbert polynomial.

Proposition 2.6.2. *Let L be a nef and big Weil divisor. If*

$$h^0(mL)>i(X)L^3m+1$$

for some integer m, then $|mL|$ is not composed with a pencil of surfaces.

Proof. Assume that $|mL|$ is composed with a pencil of surfaces. Set $D:=mL$ and keep the same notation as in Section 2.4. Then we have

$$m\pi^*(L)\geq M\equiv aS\geq(h^0(mL)-1)S.$$

Note that by Lemma 2.6.1, $i(X)\pi^*(L)^2\cdot S$ is an integer. On the other hand, $\pi^*(L)^2\cdot S$ is positive since $\pi^*(L)|_S$ is nef and big on S. Hence $\pi^*(L)^2\cdot S\geq\frac{1}{i(X)}$. Thus we have $mL^3\geq(h^0(mL)-1)(\pi^*(L)^2\cdot S)\geq\frac{1}{i(X)}(h^0(mL)-1)$, a contradiction. □

2.7 Main reduction

We reduce the birationality and generic finiteness problems on a singular X to that on its smooth model Y.

Lemma 2.7.1 (cf. [Chen11, Lemma 2.5])**.** *Let W be a normal projective variety on which there is an integral Weil $\mathbb{Q}$-Cartier divisor D. Let $h: V\longrightarrow W$ be any resolution of singularities. Assume that E is an effective exceptional $\mathbb{Q}$-divisor on V with $h^*(D)+E$ a Cartier divisor on V. Then*

$$h_*\mathcal{O}_V(h^*(D)+E)=\mathcal{O}_W(D)$$

where $\mathcal{O}_W(D)$ is the reflexive sheaf corresponding to the Weil divisor D.

Lemma 2.7.2. *Let X be a normal projective variety with $\mathbb{Q}$-factorial terminal singularities, D be a Weil divisor on X, and $\pi : Y \longrightarrow X$ be a resolution. Then $\Phi_{|K_X+D|}$ is birational (resp. generically finite) if and only if so is $\Phi_{|K_Y+\lceil\pi^*(D)\rceil|}$.*

Proof. Recall that

$$K_Y = \pi^*(K_X) + E_\pi$$

where E_π is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor since X has at worst terminal singularities. We have

$$\begin{aligned} & K_Y + \lceil\pi^*(D)\rceil \\ &= \pi^*(K_X) + E_\pi + \pi^*(D) + E \\ &= \pi^*(K_X + D) + E_\pi + E \end{aligned}$$

where $E_\pi + E$ is an effective $\mathbb{Q}$-divisor on Y exceptional over X. Lemma 2.7.1 implies

$$\pi_*\mathcal{O}_Y(K_Y + \lceil\pi^*(D)\rceil) = \mathcal{O}_X(K_X + D).$$

Hence $\Phi_{|K_X+D|}$ is birational (resp. generically finite) if and only if so is $\Phi_{|K_Y+\lceil\pi^*(D)\rceil|}$. $\square$

3 Boundedness of anti-canonical volumes of singular log Fano threefolds

In this chapter, we investigate the boundedness of the anti-canonical volume of an ϵ-klt log Fano pair of dimension three. We will prove Theorem 1.1.7.

This chapter is organized as follows. In Section 3.1, we prove the reduction step to Mori fiber spaces (Theorem 1.1.11). In Section 3.2, we prove generalized Ambro's conjecture in dimension two (Theorem 1.1.16). In Section 3.3, we prove Weak BAB Conjecture for Mori fiber spaces in dimension three (Theorem 1.1.12). In Section 3.4, we prove the boundedness of log Fano threefolds of fixed index (Corollary 1.1.8).

3.1 Reduction to Mori fiber spaces

In this section, we prove the reduction step to Mori fiber spaces (Theorem 1.1.11).

The "only if" direction is trivial, we only need to prove the "if" direction.

Fix $0 < \epsilon < 1$, an integer $n > 0$. Let (X, Δ) be an ϵ-klt log Fano pair of dimension n. By [BCHM10, Corollary 1.4.3], taking terminalization of (X, Δ) (or terminalization of X if K_X is $\mathbb{Q}$-Cartier), we have $\pi : X_1 \to X$ where $K_{X_1} + \Delta_{X_1} = \pi^*(K_X + \Delta)$, Δ_{X_1} is an effective $\mathbb{Q}$-divisor, X_1 is $\mathbb{Q}$-factorial terminal, and (X_1, Δ_{X_1}) is ϵ-klt. Here $-(K_{X_1} + \Delta_{X_1})$ is nef and big. By Kodaira's lemma (cf. [KoMo98, Proposition 2.61]) there exist a $\mathbb{Q}$-divisor Δ_1 such that $\Delta_1 \geq \Delta_{X_1}$, $-(K_{X_1} + \Delta_1)$ is ample, and (X, Δ_1) is ϵ-klt. In particular, X_1 is $\mathbb{Q}$-factorial terminal and of ϵ-Fano type.

Running K-MMP on X_1, we get a sequence of normal projective varieties:

$$X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots \dashrightarrow X_r \to T.$$

Since $-K_{X_1}$ is big, this sequence ends up with a Mori fiber space $X_r \to T$ (cf. [BCHM10, Corollary 1.3.3]). In particular, X_r is $\mathbb{Q}$-factorial terminal.

Being of ϵ-Fano type is preserved by MMP according to the following lemma.

Lemma 3.1.1 (cf. [GOST15, Lemma 3.1]). *Let Y be a projective normal variety and $f : Y \to Z$ be a projective birational contraction.*

(1) If Y is of ϵ-Fano type, so is Z;

(2) Assume that f is small, then Y is of ϵ-Fano type if and only if so is Z.

In particular, minimal model program preserves ϵ-Fano type.

Proof. The proof is almost the same as [GOST15, Lemma 3.1] where 0-Fano type is considered. First we assume that Y is of ϵ-Fano type, that is, there exists an effective $\mathbb{Q}$-divisor Δ on Y such that (Y, Δ) is ϵ-klt log Fano pair. Let H be a general effective ample divisor on Z and take a sufficiently small rational number $\delta > 0$ such that $-(K_Y + \Delta + \delta f^*H)$ is ample and $(Y, \Delta + \delta f^*H)$ is ϵ-klt. Then take a general effective ample $\mathbb{Q}$-divisor A on Y such that $(Y, \Delta + \delta f^*H + A)$ is ϵ-klt and

$$K_Y + \Delta + \delta f^*H + A \sim_{\mathbb{Q}} 0.$$

Then

$$K_Z + f_*\Delta + \delta H + f_*A = f_*(K_Y + \Delta + \delta f^*H + A) \sim_{\mathbb{Q}} 0,$$

and

$$f^*(K_Z + f_*\Delta + \delta H + f_*A) = K_Y + \Delta + \delta f^*H + A.$$

Therefore, $(Z, f_*\Delta + \delta H + f_*A)$ is ϵ-klt. Hence $(Z, f_*\Delta + f_*A)$ is ϵ-klt and $-(K_Z + f_*\Delta + f_*A) \sim_{\mathbb{Q}} \delta H$ is ample, that is, Z is of ϵ-Fano type.

Next we assume that f is small and Z is of ϵ-Fano type. Let Γ be an effective $\mathbb{Q}$-divisor on Z such that (Z, Γ) is ϵ-klt log Fano pair. Let Γ_Y be the strict transform of Γ on Y. Since f is small,

$$K_Y + \Gamma_Y = f^*(Z + \Gamma).$$

Hence (Y, Γ_Y) is ϵ-klt and $-(K_Y + \Gamma_Y)$ is nef and big. By Kodaira's lemma, there exist a $\mathbb{Q}$-divisor Γ' such that $\Gamma' \geq \Gamma_Y$, $-(K_Y + \Gamma')$ is ample, and (Y, Γ') is ϵ-klt, that is, Y is of ϵ-Fano type.

We proved the lemma. $\square$

By Lemma 3.1.1, for all i, X_i is of ϵ-Fano type. To compare the volumes between these varieties, we have the following lemma.

Lemma 3.1.2. *Let $X_i \dashrightarrow X_{i+1}$ be one step of K-MMP. Then*

$$\mathrm{Vol}(-K_{X_i}) \leq \mathrm{Vol}(-K_{X_{i+1}}).$$

Proof. Take a common resolution $p : W \to X_i$, $q : W \to X_{i+1}$. Then

$$p^*(K_{X_i}) = q^*(K_{X_{i+1}}) + E,$$

where E is an effective q-exceptional $\mathbb{Q}$-divisor. Hence

$$\begin{aligned}\mathrm{Vol}(-K_{X_i}) &= \mathrm{Vol}(-p^*(K_{X_i})) \\ &= \mathrm{Vol}(-q^*(K_{X_{i+1}}) - E) \\ &\leq \mathrm{Vol}(-q^*(K_{X_{i+1}})) \\ &= \mathrm{Vol}(-K_{X_{i+1}}).\end{aligned}$$

We proved the lemma. $\square$

Therefore we can compare the volumes on X and X_r by Lemma 3.1.2:

$$\begin{aligned}(-(K_X + \Delta))^n &= (-(K_{X_1} + \Delta_{X_1}))^n \\ &= \mathrm{Vol}(-(K_{X_1} + \Delta_{X_1})) \\ &\leq \mathrm{Vol}(-K_{X_1}) \\ &\leq \mathrm{Vol}(-K_{X_r}).\end{aligned}$$

Now X_r is an n-dimensional ϵ-Fano type variety with a Mori fiber structure by construction. Assuming Weak BAB Conjecture for Mori fiber spaces, there exists $M(n, \epsilon)$ such that

$$\mathrm{Vol}(-K_{X_r}) \leq M(n, \epsilon).$$

Hence

$$(-(K_X + \Delta))^n \leq M(n, \epsilon).$$

Moreover, if K_X is $\mathbb{Q}$-Cartier, then since we take X_1 as the terminalization of X, we have $K_{X_1} + F = \pi^* K_X$ with F an effective $\mathbb{Q}$-divisor. Hence

$$\mathrm{Vol}(-K_X) \leq \mathrm{Vol}(-K_{X_1}) \leq \mathrm{Vol}(-K_{X_r}) \leq M(n, \epsilon).$$

We complete the proof of Theorem 1.1.11.

As a direct corollary, we recover the main result in [Jiang13] on Weak BAB Conjecture in dimension two.

Corollary 3.1.3. *Fix $0 < \epsilon < 1$.*

Then there exists a number

$$M(2, \epsilon) := \max \left\{ 9, \lfloor 2/\epsilon \rfloor + 4 + \frac{4}{\lfloor 2/\epsilon \rfloor} \right\}$$

with the following property:

If (X, Δ) is an ϵ-klt log del Pezzo pair, then

$$(K_X + \Delta)^2 \leq M(2, \epsilon).$$

Further, if K_X is $\mathbb{Q}$-Cartier, then

$$\mathrm{Vol}(-K_X) \leq M(2, \epsilon).$$

Proof. By Theorem 1.1.11, we only need to consider the cases when $X = \mathbb{P}^2$ or $\mathbb{F}_n$ with $n \leq 2/\epsilon$ (see [AM04, Lemma 1.4] or [Jiang13, Lemma 3.1]). And the result follows by volume computation directly. □

3.2 Generalized Ambro's conjecture in dimension two

In this section, we prove generalized Ambro's conjecture in dimension two (Theorem 1.1.16).

Fix an ϵ-klt weak log del Pezzo pair (S, B) with S smooth and a $\mathbb{Q}$-divisor $G \sim_{\mathbb{Q}} -(K_S + B)$ such that $G + B \geq 0$. Set $a := \mathrm{glct}(S, B; G)$. Since we work on $\mathbb{Q}$-divisors, a is a positive rational number. The problem is to bound a from below. We may assume that $a < 1$. Set $D = G + B \geq 0$. Then $(S, B + aG) = (S, (1-a)B + aD)$ is not klt. Note that $D \sim_{\mathbb{Q}} -K_S$.

By Base Point Free Theorem (cf. [KoMo98, Theorem 3.3]), $-(K_S + B)$ is semi-ample. Hence there exists an effective $\mathbb{Q}$-divisor M such that $K_S + B + M \sim_{\mathbb{Q}} 0$ and $(S, B + M)$ is ϵ-klt. For any birational morphism $f : S \to T$ between smooth surfaces, we have

$$K_S + B + M = f^*(K_T + f_*B + f_*M),$$
$$K_S + (1-a)(B + M) + aD = f^*(K_T + (1-a)(f_*B + f_*M) + af_*D).$$

Hence $(T, f_*B + f_*M)$ is ϵ-klt and $(T, (1-a)(f_*B + f_*M) + af_*D)$ is not klt with

$$K_T + f_*B + f_*M \sim_{\mathbb{Q}} K_T + (1-a)(f_*B + f_*M) + af_*D \sim_{\mathbb{Q}} 0.$$

Recall that either $S \simeq \mathbb{P}^2$ or there exists a birational morphism $g : S \to \mathbb{F}_n$ with $n \le 2/\epsilon$ by [AM04, Lemma 1.4] or [Jiang13, Lemma 3.1].

Hence by replacing S by $T = \mathbb{P}^2$ or $\mathbb{F}_n$, we may assume that there exists a triple (T, B_T, D_T) satisfying the following conditions:

(i) $T = \mathbb{P}^2$ or $\mathbb{F}_n$ with $n \le 2/\epsilon$;

(ii) B_T, D_T are effective $\mathbb{Q}$-divisors on T;

(iii) (T, B_T) is ϵ-klt and $(T, (1-a)B_T + aD_T)$ is not klt;

(iv) $K_T + B_T \sim_{\mathbb{Q}} K_T + (1-a)B_T + aD_T \sim_{\mathbb{Q}} 0$, equivalently, $B_T \sim_{\mathbb{Q}} D_T \sim_{\mathbb{Q}} -K_T$.

Since $(T, (1-a)B_T + aD_T)$ is not klt, we may take a sequence of point blow-ups

$$T_{r+1} \to T_r \to \cdots \to T_2 \to T_1 = T$$

where $T_{i+1} \to T_i$ is the blow-up at a non-klt center $P_i \in \mathrm{Nklt}(T_i, (1-a)B_i + aD_i + E_i)$ where B_i and D_i are the strict transforms of B_T and D_T respectively and

$$K_{T_i} + (1-a)B_i + aD_i + E_i = \pi_i^*(K_T + (1-a)B_T + aD_T),$$

where $\pi_i : T_i \to T$ is the composition map and E_i is a π_i-exceptional $\mathbb{Q}$-divisor. We stop this process at T_{r+1} if

$$\dim \mathrm{Nklt}(T_{r+1}, (1-a)B_{r+1} + aD_{r+1} + E_{r+1}) > 0.$$

Since P_i is a non-klt center of $(T_i, (1-a)B_i + aD_i + E_i)$, $\mathrm{mult}_{P_i}((1-a)B_i + aD_i + E_i) \ge 1$. Note that the coefficients of E_i are $(\mathrm{mult}_{P_j}((1-a)B_j + aD_j + E_j) - 1)$ for $j < i$, hence E_i is effective for all i. Furthermore, we may assume that $\mathrm{mult}_{P_i} B_i$ is non-increasing. Take the integer $k \le r$ such that $\mathrm{mult}_{P_i} B_i \ge \epsilon/2$ for $i \le k$ and $\mathrm{mult}_{P_i} B_i < \epsilon/2$ for $i > k$. Write $B_T = \sum_j b_j B^j$ and $B_i = \sum_j b_j B_i^j$ by components. We have $b_j < 1 - \epsilon$ since (T, B_T) is ϵ-klt. Recall that $\sum_j b_j \le 4$ by Lemma 2.2.2.

Claim 1. *If* $\mathrm{mult}_{B^j}(aD_T) > \epsilon/2$ *for some* j, *then* $a \ge \epsilon^2/(4 + 4\epsilon)$.

Proof. Recall that $T = \mathbb{P}^2$ or $\mathbb{F}_n$ with $n \le 2/\epsilon$.

If $T = \mathbb{P}^2$, then $\mathrm{mult}_{B^j} D_T \le 3$ be degree counting. If $T = \mathbb{F}_n$ and B^j is a fiber, then $\mathrm{mult}_{B^j} D_T \le n + 2 \le 2/\epsilon + 2$ by Lemma 2.2.1. If $T = \mathbb{F}_n$ and B^j is not a fiber, then $\mathrm{mult}_{B^j} D_T \le D_T \cdot f = 2$ where f is a fiber. Hence

$$a \ge \frac{\epsilon}{2\mathrm{mult}_{B^j} D_T} \ge \frac{\epsilon^2}{4 + 4\epsilon}.$$

We proved the claim. $\square$

Since we need a lower bound of a, from now on, we may assume that $\mathrm{mult}_{B^j}(aD_T) \leq \epsilon/2$ for all j. In particular, $\mathrm{mult}_{B_i^j}(aD_i) \leq \epsilon/2$ and

$$\mathrm{mult}_{B_i^j}((1-a)B_i + aD_i) < 1 - \epsilon/2$$

for all i and j.

Claim 2. *$(B_{k+1}^j)^2 \geq -4/\epsilon$ for all j.*

Proof. If $(B_{k+1}^j)^2 < 0$, then

$$\begin{aligned}
-2 &\leq 2p_a(B_{k+1}^j) - 2 = (K_{T_{k+1}} + B_{k+1}^j) \cdot B_{k+1}^j \\
&= \frac{\epsilon}{2}(B_{k+1}^j)^2 + (K_{T_{k+1}} + (1-\frac{\epsilon}{2})B_{k+1}^j) \cdot B_{k+1}^j \\
&\leq \frac{\epsilon}{2}(B_{k+1}^j)^2 + (K_{T_{k+1}} + (1-a)B_{k+1} + aD_{k+1} + E_{k+1}) \cdot B_{k+1}^j \\
&= \frac{\epsilon}{2}(B_{k+1}^j)^2 < 0,
\end{aligned}$$

Hence we proved the claim. $\square$

Now we can bound the number k. On T_{k+1}, we have

$$\begin{aligned}
(B_{k+1})^2 &= (\sum_j b_j B_{k+1}^j)^2 \geq \sum_j b_j^2 (B_{k+1}^j)^2 \geq (\sum_j b_j^2)(-4/\epsilon) \\
&\geq (\sum_j b_j)(1-\epsilon)(-4/\epsilon) \geq 16 - \frac{16}{\epsilon}
\end{aligned}$$

and $(B_1)^2 = (K_T)^2 \leq 9$. On the other hand, at each blow-up, $(B_i)^2$ decreases by at least $\epsilon^2/4$ by the assumption $\mathrm{mult}_{P_i} B_i \geq \epsilon/2$ for $i \leq k$. Hence

$$k \leq \frac{9 - (16 - 16/\epsilon)}{\epsilon^2/4} \leq \frac{64}{\epsilon^3}.$$

Now we consider $\pi_{k+1}^*(aD_T)$ on T_{k+1}.

Claim 3. *There exists a point Q on T_{k+1} such that $\mathrm{mult}_Q \pi_{k+1}^*(aD_T) \geq \epsilon/4$.*

Proof. Consider the pair $(T_{k+1}, (1-a)B_{k+1} + aD_{k+1} + E_{k+1})$. Note that E_{k+1} is simple normal crossing supported.

Assume that there exists a curve E with coefficient at least $1 - 3\epsilon/4$ in E_{k+1}, that is,

$$\mathrm{mult}_E(K_{T_{k+1}} - \pi_{k+1}^*(K_T + (1-a)B_T + aD_T)) \leq -1 + 3\epsilon/4.$$

On the other hand, since $(T, (1-a)B_T)$ is ϵ-klt,

$$\mathrm{mult}_E(K_{T_{k+1}} - \pi_{k+1}^*(K_T + (1-a)B_T)) \geq -1 + \epsilon.$$

Hence $\mathrm{mult}_E \pi_{k+1}^*(aD_T) \geq \epsilon/4$.

If all coefficients of E_{k+1} are smaller than $1 - 3\epsilon/4$, then $k < r$ and P_{k+1} is a non-klt center of $(T_{k+1}, (1-a)B_{k+1} + aD_{k+1} + E_{k+1})$. By Lemma 2.3.4, $\mathrm{mult}_{P_{k+1}}((1-a)B_{k+1} + aD_{k+1}) \geq 3\epsilon/4$. Then $\mathrm{mult}_{P_{k+1}}(aD_{k+1}) \geq \epsilon/4$ since $\mathrm{mult}_{P_{k+1}} B_{k+1} < \epsilon/2$ by assumption. In particular, $\mathrm{mult}_{P_{k+1}} \pi_{k+1}^*(aD_T) \geq \mathrm{mult}_{P_{k+1}}(aD_{k+1}) \geq \epsilon/4$.

We proved the claim. □

Now we will estimate $\mathrm{mult}_{Q_1}(aD_T)$ where Q_1 is the image of Q on T. By removing unnecessary blow-ups, we may assume that we have a sequence of blow-ups

$$T_{k+1} \to T_k \to \cdots \to T_2 \to T_1 = T$$

where $f_{i+1} : T_{i+1} \to T_i$ is the blow-up at Q_i which is the image of Q on T_i with $k \leq 64/\epsilon^3$. Recall that $\pi_i : T_i \to T$ is the composition map and D_i is the strict transform of D_T on T_i. Denote C_{i+1} to be the exceptional divisor of f_{i+1} and C_{i+1}^j be its strict transform on T_j for $j \geq i+1$. We can write

$$\pi_j^*(aD_T) = aD_j + \sum_{2 \leq i \leq j} c_i C_i^j,$$

with $c_i = \mathrm{mult}_{Q_{i-1}} \pi_{i-1}^*(aD_T)$.

Claim 4. *If* $\mathrm{mult}_{Q_1}(aD_T) \leq \alpha$, *then* $\mathrm{mult}_{Q_i} \pi_i^*(aD_T) \leq (\mathsf{F}_{i+1} - 1)\alpha$ *for* $1 \leq i \leq k+1$. *Here* F_n *is the Fibonacci number with relation* $\mathsf{F}_n = \mathsf{F}_{n-1} + \mathsf{F}_{n-2}$ *for all* $n \geq 2$ *and* $\mathsf{F}_0 = \mathsf{F}_1 = 1$.

Proof. We run induction on i. The case $i = 1$ is trivial. Assume the conclusion holds for $i < j$, then noting that Q_j is contained in at most two exceptional curves, we have

$$\begin{aligned}
\mathrm{mult}_{Q_j} \pi_j^*(aD_T) &= \mathrm{mult}_{Q_j}(aD_j + \sum_{2 \leq i \leq j} c_i C_i^j) \\
&\leq \mathrm{mult}_{Q_j}(aD_j) + (\mathsf{F}_j - 1)\alpha + (\mathsf{F}_{j-1} - 1)\alpha \\
&\leq \mathrm{mult}_{Q_1}(aD_1) + (\mathsf{F}_j - 1)\alpha + (\mathsf{F}_{j-1} - 1)\alpha \\
&\leq (\mathsf{F}_{j+1} - 1)\alpha.
\end{aligned}$$

We proved the claim. □

By Claims 3 and 4, $\mathrm{mult}_{Q_1}(aD_T) \geq \epsilon/(4\mathsf{F}_{k+2} - 4)$. Recall that $D_T \sim_{\mathbb{Q}} -K_T$ and $T = \mathbb{P}^2$ or $\mathbb{F}_n$ with $n \leq 2/\epsilon$. By Lemma 2.2.2, $\mathrm{mult}_{Q_1}(D_T) \leq n+4$, combining with the inequality $k \leq 64/\epsilon^3$, we have

$$a \geq \frac{\epsilon^2}{(2+4\epsilon)(4\mathsf{F}_{\lfloor 64/\epsilon^3 \rfloor + 2} - 4)},$$

and hence we may take this number to be $\mu(2, \epsilon)$.

We have proved Theorem 1.1.16.

3.3 Weak BAB Conjecture for Mori fiber spaces in dimension three

In this section, we prove Weak BAB Conjecture for Mori fiber spaces in dimension 3 (Theorem 1.1.12). Recall that by a Mori fiber space we always mean a $\mathbb{Q}$-factorial terminal one.

Fix $0 < \epsilon < 1$ and consider an ϵ-klt log Fano pair (X, Δ) of dimension 3 with a Mori fiber structure. As explained, there are three cases:

(1) X is a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-folds with $\rho = 1$;

(2) $X \to \mathbb{P}^1$ is a del Pezzo fibration;

(3) $X \to S$ is a conic bundle.

As mentioned before, Case (1) is done by Kawamata [Kaw92a]. We treat Cases (2) and (3) in the following two subsections.

3.3.1 Contractions to a curve

In this subsection, we treat the case under a more general setting when there is a contraction $f : X \to \mathbb{P}^1$ (e.g. X has a del Pezzo fibration structure).

Theorem 3.3.1. *Let (Y, B) be an ϵ-klt log Fano pair of dimension n with a contraction $g : Y \to \mathbb{P}^1$ and Y having terminal singularities. Assume that Weak BAB Conjecture and generalized Ambro's conjecture hold in dimension $n-1$ with $M(n-1, \epsilon)$ and $\mu(n-1, \epsilon)$ the numbers defined in these conjectures. Then*

$$\mathrm{Vol}(-K_Y) \leq \frac{2nM(n-1,\epsilon)}{\mu(n-1,\epsilon)}.$$

Proof. Note that Y is terminal by assumption. Hence a general fiber F of g is terminal and of ϵ-Fano type of dimension $n-1$ by adjunction formula. In particular, K_F is $\mathbb{Q}$-Cartier. It follows that $\mathrm{Vol}(-K_F) \leq M(n-1,\epsilon)$ by Weak BAB Conjecture in dimension $n-1$.

Fix a rational number s satisfying

$$\frac{\mathrm{Vol}(-K_Y)}{nM(n-1,\epsilon)} - \frac{1}{A} < s < \frac{\mathrm{Vol}(-K_Y)}{nM(n-1,\epsilon)}$$

for sufficiently large number A. To bound $\mathrm{Vol}(-K_Y)$ from above, it is sufficient to bound s from above. And we may assume that $s > 2$.

The following lemma allows us to construct non-klt centers.

Claim 5. *For a general fiber F of g, $-K_Y - sF$ is $\mathbb{Q}$-effective. In particular, there exists an effective $\mathbb{Q}$-divisor $B_F \sim_{\mathbb{Q}} -\frac{1}{s}K_Y$ such that F is a non-klt center of (Y, B_F).*

Proof. For a positive integer p and a sufficiently divisible positive integer m, we have exact sequence

$$0 \to \mathcal{O}_Y(-mK_Y - pF) \to \mathcal{O}_Y(-mK_Y-(p-1)F) \to \mathcal{O}_F(-mK_Y-(p-1)F) \to 0.$$

Note that $\mathcal{O}_F(-mK_Y-(p-1)F) = \mathcal{O}_F(-mK_F)$. Hence

$$h^0(Y, \mathcal{O}_Y(-mK_Y - pF)) \geq h^0(Y, \mathcal{O}_Y(-mK_Y-(p-1)F)) - h^0(F, \mathcal{O}_F(-mK_F)).$$

Inductively, we have

$$h^0(Y, \mathcal{O}_Y(-mK_Y - pF)) \geq h^0(Y, \mathcal{O}_Y(-mK_Y)) - ph^0(F, \mathcal{O}_F(-mK_F)).$$

We may take $p = sm$ since m is sufficiently divisible. By the definition of volume, we have

$$\begin{aligned}&\limsup_{m\to\infty} \frac{h^0(Y, \mathcal{O}_Y(-mK_Y)) - smh^0(F, \mathcal{O}_F(-mK_F))}{m^n} \\ &= \frac{1}{n!}\mathrm{Vol}(-K_Y) - \frac{s}{(n-1)!}\mathrm{Vol}(-K_F) > 0.\end{aligned}$$

Hence $h^0(Y, \mathcal{O}_Y(-mK_Y - smF)) > 0$ for m sufficiently divisible, that is, $-K_Y - sF$ is $\mathbb{Q}$-effective. In particular, there exists an effective $\mathbb{Q}$-divisor $B_F \sim_{\mathbb{Q}} -\frac{1}{s}K_Y$ such that $B_F - F \geq 0$, and hence F is a non-klt center of (Y, B_F).

We proved the claim. $\square$

Now for two general fibers F_1 and F_2, consider $B' = B_{F_1} + B_{F_2}$. By construction, $F_1 \cup F_2 \subset \mathrm{Nklt}(Y, (1-\frac{2}{s})B + B')$. Note that

$$-(K_Y + (1-\frac{2}{s})B + B') \sim_{\mathbb{Q}} -(1-\frac{2}{s})(K_Y + B)$$

is ample, by Connectedness Lemma, $\mathrm{Nklt}(Y, (1-\frac{2}{s})B + B')$ is connected. Hence there is a non-klt center $W \subset \mathrm{Nklt}(Y, (1-\frac{2}{s})B + B')$ connecting F_1 and F_2. In particular, W dominates $\mathbb{P}^1$. Restricting on a general fiber F, by adjunction formula, we have $(F, B|_F)$ is ϵ-klt log Fano with F terminal and $(F, (1-\frac{2}{s})B|_F + B'|_F)$ is not klt (see [KoMo98, Lemma 5.17, Lemma 5.50]) with $B'|_F \sim_{\mathbb{Q}} -\frac{2}{s}K_F$. Hence

$$\frac{2}{s} \geq \mathrm{glct}(F, B|_F; \frac{s}{2}B'|_F - B|_F).$$

To bound s from above, generalized Ambro's conjecture arises naturally. By generalized Ambro's conjecture in dimension $n-1$,

$$s \leq \frac{2}{\mu(n-1,\epsilon)},$$

and hence

$$\mathrm{Vol}(-K_Y) \leq \frac{2nM(n-1,\epsilon)}{\mu(n-1,\epsilon)}.$$

We completed the proof. □

In particular, by Corollary 3.1.3 and Theorem 1.1.16, Weak BAB Conjecture and generalized Ambro's conjecture hold in dimension 2, and hence the following corollary holds.

Corollary 3.3.2. *Let (X, Δ) be an ϵ-klt log Fano pair of dimension 3 with a contraction $f : X \to \mathbb{P}^1$ and X having terminal singularities. Then*

$$\mathrm{Vol}(-K_X) \leq \frac{6M(2,\epsilon)}{\mu(2,\epsilon)}.$$

3.3.2 Conic bundles

In this subsection, we treat the case that X has a conic bundle structure $f : X \to S$. Firstly we collect some facts about singularities of the surface S.

Theorem 3.3.3. *Let (X, Δ) be an ϵ-klt log Fano pair of dimension 3 and $f : X \to S$ be a Mori fiber space to a surface S, then* *(i)* *S has only Du Val singularities of type A;*

(ii) *There exists an effective* $\mathbb{Q}$*-divisor* Δ_S *on* S*, such that* (S, Δ_S) *is klt log del Pezzo;*

(iii) *S is a Mori dream space;*

(iv) *There exists an effective* $\mathbb{Q}$*-divisor* Δ'_S *on* S*, such that* (S, Δ'_S) *is* $\delta(\epsilon)$*-klt and* $K_S + \Delta'_S \sim_{\mathbb{Q}} 0$*, where* $\delta(\epsilon)$ *is a positive real number depending only on* ϵ*;*

(v) *The family of such S is bounded, in particular, the Picard number of minimal resolution of S is bounded by* $128/\delta(\epsilon)^5$*.*

(vi) *S is* $N(\epsilon)$*-factorial, i.e. for a Weil divisor D on S,* $N(\epsilon)D$ *is Cartier, where* $N(\epsilon)$ *is a positive integer depending only on* ϵ*.*

Proof. (i) is by [MP08, (1.2.7) Theorem]. (ii) is by [FG12, Corollary 3.3]. (iii) is by (ii) and [BCHM10, Corollary 1.3.2]. (iv) is by [Bir14, Corollary 1.7] since we may find a boundary $\Delta' \geq \Delta$ such that (X, Δ') is ϵ-klt and $K_X + \Delta' \sim_{\mathbb{Q}} 0$. (v) is by (iv), [Ale94a, Theorem 6.8], and [AM04, Theorem 1.8]. (vi) is a direct consequence of (i) and (v). $\square$

For the definition and properties of *Mori dream spaces* we refer to the famous paper by Hu–Keel [HK00]. We will use the following property of Mori dream spaces: every nef divisor on S is semi-ample and there are finitely many irreducible curves with negative self intersection. In particular, a curve through a general point is nef. By a curve we always mean an irreducible reduced one.

If there is a curve C on S satisfying $(C)^2 = 0$, then C is semi-ample. In particular, a multiple of C induces a contraction $S \to \mathbb{P}^1$ and we are done by Subsection 3.3.1. Hence we may assume that there does not exist such curve C on S satisfying $(C)^2 = 0$.

Fix a positive rational number t satisfying

$$\frac{\mathrm{Vol}(-K_X)}{24} - \frac{1}{A} < t^2 < \frac{\mathrm{Vol}(-K_X)}{24}$$

for sufficiently large number A. To bound $\mathrm{Vol}(-K_X)$ from above, it is sufficient to bound t from above. And we may assume that $t > 768N(\epsilon)/\epsilon$.

Lemma 3.3.4. *For a general fiber* F *of* f*,*

$$h^0(X, \mathcal{O}_X(-mK_X) \otimes \mathcal{I}_F^{2tm}) > 0$$

for m *sufficiently divisible, where* $\mathcal{I}_F$ *is the ideal sheaf of* F*. In particular, there exists an effective* $\mathbb{Q}$*-divisor* $\Delta_F \sim_{\mathbb{Q}} -\frac{1}{t}K_X$ *such that* $\mathrm{mult}_F \Delta_F \geq 2$*.*

Proof. For a positive integer p and a sufficiently divisible positive integer m, we have exact sequence

$$0 \to \mathcal{O}_X(-mK_X)\otimes\mathcal{I}_F^p \to \mathcal{O}_X(-mK_X)\otimes\mathcal{I}_F^{p-1} \to \mathcal{O}_X(-mK_X)\otimes\mathcal{I}_F^{p-1}/\mathcal{I}_F^p \to 0.$$

Note that $\mathcal{I}_F^{p-1}/\mathcal{I}_F^p = S^{p-1}(\mathcal{I}_F/\mathcal{I}_F^2)$ (see [Har77, II. Theorem 8.24]) and $\mathcal{I}_F/\mathcal{I}_F^2 = \mathcal{O}_F^{\oplus 2}$. Hence

$$h^0(X, \mathcal{O}_X(-mK_X)\otimes\mathcal{I}_F^p) \geq h^0(X, \mathcal{O}_X(-mK_X)\otimes\mathcal{I}_F^{p-1}) - ph^0(F, \mathcal{O}_F(-mK_F)).$$

Inductively, we have

$$h^0(X, \mathcal{O}_X(-mK_X)\otimes\mathcal{I}_F^p) \geq h^0(X, \mathcal{O}_X(-mK_X)) - \frac{p(p+1)}{2}h^0(F, \mathcal{O}_F(-mK_F)).$$

We may take $p = 2tm$ since m is sufficiently divisible. By the definition of volume, we have

$$\begin{aligned} &\limsup_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(-mK_X)) - (2t^2m^2 + tm)h^0(F, \mathcal{O}_F(-mK_F))}{m^3} \\ =& \frac{1}{6}\mathrm{Vol}(-K_X) - 2t^2\mathrm{Vol}(-K_F) > 0. \end{aligned}$$

Note that $F \simeq \mathbb{P}^1$ and $\mathrm{Vol}(-K_F) = 2$. Hence $h^0(X, \mathcal{O}_X(-mK_X)\otimes\mathcal{I}_F^{2tm}) > 0$ for m sufficiently divisible. In particular, there exists an effective $\mathbb{Q}$-divisor $\Delta_F \sim_{\mathbb{Q}} -\frac{1}{t}K_X$ such that $\mathrm{mult}_F\Delta_F \geq 2$. □

A prime divisor V on X is *vertical* if $f(V)$ is a curve or V does not dominate S. Note that for a curve C on S passing through a general point, there is only one vertical prime divisor contained in $f^{-1}(C)$ and we denote it by V_C. It is easy to see that $f^*C = V_C$ as Weil divisors is well defined (since removing finitely many points of S, f is flat). For a general point $P \in S$, denote F_P be the fiber at P.

We can modify the $\mathbb{Q}$-divisor Δ_F to control vertical divisors by the following lemma.

Lemma 3.3.5. *For a general point $P \in S$, there exist an effective $\mathbb{Q}$-divisor $B_P \sim_{\mathbb{Q}} -\frac{a_P}{t}K_X$ for some $a_P \leq 384N(\epsilon)/\epsilon$ such that*

(i) $\mathrm{mult}_{F_P}B_P \geq 2$, *and*

(ii) For any curve C passing through P, $\mathrm{mult}_{V_C}B_P \leq \epsilon/2$.

Proof. Write $\Delta_{F_P} = \Delta_0 + \sum_i b_i V_{C_i}$ where Δ_0 does not contain vertical divisors passing through F_P and C_i is a curve passing through P. If $b_i \le \epsilon/2$ for all i, then we can take $B_P = \Delta_{F_P}$ with $a_P = 1$.

Now we assume that $b_1 > \epsilon/2$. Note that by assumption, $(C_1)^2 > 0$ and $N(\epsilon)C_1$ is Cartier. So $N(\epsilon)C_1$ is a nef and big Cartier divisor. Hence by Kollár's Effective Base Point Free Theorem (see [Kol93, 1.1 Theorem]), $|96N(\epsilon)C_1|$ is base point free. It follows that $|96N(\epsilon)C_1|$ defines a generically finite map $\Phi : S \to \mathbb{P}(|96N(\epsilon)C_1|)$. Since $P \in C_1$, P is not on the contracted curves of Φ. Hence by taking the pull back of general hyperplanes passing through $\Phi(P)$, we can write $96N(\epsilon)C_1 \sim_{\mathbb{Q}} \sum_j h_j H_j$ with $96N(\epsilon)C_1 \sim H_j$ a general curve passing through P, $0 < h_j \le \epsilon/4$ for all j, and $\sum_j h_j = 1$. For $i > 1$, since C_i is semi-ample, we may take $C_i \sim_{\mathbb{Q}} D_i$ such that D_i is an effective $\mathbb{Q}$-divisor on S passing through a general point but not containing P. Now define

$$B_P := \frac{192N(\epsilon)}{b_1}\Big(\Delta_0 + \sum_{i>1} b_i f^* D_i\Big) + 2\sum_j h_j f^* H_j \sim_{\mathbb{Q}} -\frac{192N(\epsilon)}{b_1 t} K_X,$$

and we can take $a_P = 192N(\epsilon)/b_1 \le 384N(\epsilon)/\epsilon$. Note that

$$\mathrm{mult}_{F_P} B_P \ge \mathrm{mult}_P\big(2\sum_j h_j H_j\big) \ge 2.$$

And by construction, for every curve C passing through P, if $C = H_j$ for some j, then $\mathrm{mult}_{V_C} B_P = 2h_j \le \epsilon/2$; otherwise $\mathrm{mult}_{V_C} B_P = 0$.

Hence we proved the lemma. $\square$

Take two general points P_1 and P_2 on S. For simplicity, for $i = 1, 2$, we denote $F_{P_i} = F_i$, $a_{P_i} = a_i$, and $B_{P_i} = B_i$. Note that by construction,

$$F_i \subset \mathrm{Nklt}(X, B_i).$$

Case 1. There exists a non-klt center E of dimension 2 of (X, B_i) containing F_i for some $i = 1$ or 2.

In this case,

$$\mathrm{mult}_E(B_i) \ge 1.$$

By construction of B_i, E is not vertical. Restricting on a general fiber F of f, we have

$$\frac{2a_i}{t} = -\frac{a_i}{t} K_X \cdot F = B_i \cdot F \ge E \cdot F \ge 1.$$

Hence

$$t \le 2a_i \le \frac{768N(\epsilon)}{\epsilon}.$$

Case 2. F_i is a maximal non-klt center of (X, B_i) for $i = 1$ and 2.

Since P_1 is a general point, we may assume $F_1 \not\subset \operatorname{Supp}(\Delta + B_2)$. Hence F_1 is a maximal non-klt center of $(X, (1 - \frac{a_1+a_2}{t})\Delta + B_1 + B_2)$ and F_2 is a non-klt center. Note that

$$-(K_X + (1 - \frac{a_1 + a_2}{t})\Delta + B_1 + B_2) \sim_{\mathbb{Q}} -(1 - \frac{a_1 + a_2}{t})(K_X + \Delta)$$

is ample by the assumption $t > 768N(\epsilon)/\epsilon$. By Connectedness Lemma, $\operatorname{Nklt}(X, (1 - \frac{a_1+a_2}{t})\Delta + B_1 + B_2)$ is connected. Hence there is a non-klt center W intersecting with F_1. Hence we have

$$\operatorname{mult}_W\Big(\Big(1 - \frac{a_1 + a_2}{t}\Big)\Delta + B_1 + B_2\Big) \geq 1.$$

If $\dim W = 2$, since $(X, (1 - \frac{a_1+a_2}{t})\Delta)$ is ϵ-klt,

$$\operatorname{mult}_W\Big(\Big(1 - \frac{a_1 + a_2}{t}\Big)\Delta\Big) < 1 - \epsilon.$$

Hence

$$\operatorname{mult}_W(B_1 + B_2) \geq \epsilon.$$

Since $F_1 \not\subset W$ by the maximality of F_1, W is not vertical. Restricting on a general fiber F of f, we have

$$\frac{2(a_1 + a_2)}{t} = -\frac{a_1 + a_2}{t} K_X \cdot F = (B_1 + B_2) \cdot F \geq \epsilon W \cdot F \geq \epsilon.$$

Hence

$$t \leq \frac{2(a_1 + a_2)}{\epsilon} \leq \frac{1536N(\epsilon)}{\epsilon^2}.$$

If $\dim W = 1$, then since P_1 is general, we may assume that for each point $Q \in F_1 \cap \operatorname{Supp}(\Delta + B_2)$, Q is not contained in the singular locus of $\operatorname{Supp}(\Delta + B_2)$. This is because the singular locus of $\operatorname{Supp}(\Delta + B_2)$ has dimension at most 1 and hence does not dominate S. Now if $W \subset \operatorname{Supp}\Delta$, then W is contained in exactly one component of Δ since $\operatorname{Supp}\Delta$ is smooth at points in $F_1 \cap W$. Since (X, Δ) is ϵ-klt, the coefficients of Δ is smaller than $1 - \epsilon$. Hence

$$\operatorname{mult}_W \Delta \leq 1 - \epsilon.$$

Of course this inequality also holds if $W \not\subset \operatorname{Supp}\Delta$. So we have

$$\operatorname{mult}_W(B_1 + B_2) \geq \epsilon.$$

Note that to compute the intersection number $(B_1 + B_2) \cdot F$ for some fiber F by $\operatorname{mult}_W(B_1 + B_2)$, it is necessary to avoid $V_{f(W)}$ in $B_1 + B_2$. Denote $V_{f(W)}$ by

V. By construction of B_1, $\mathrm{mult}_V B_1 \leq \epsilon/2$. On the other hand, $\mathrm{mult}_V B_2 = 0$ since $F_1 \not\subset \mathrm{Supp} B_2$ but $F_1 \subset V$. We can write $B_1 + B_2 = B + \lambda V$ where the support of B does not contain V. Then $\lambda \leq \epsilon/2$. It is easy to see that $\mathrm{mult}_W V = 1$ and hence

$$\mathrm{mult}_W B = \mathrm{mult}_W (B_1 + B_2) - \lambda \geq \frac{\epsilon}{2}.$$

Restricting on a fiber F of f at a general point of $f(W)$, we have

$$\frac{2(a_1 + a_2)}{t} = -\frac{a_1 + a_2}{t} K_X \cdot F = (B_1 + B_2) \cdot F = B \cdot F \geq \frac{\epsilon}{2}.$$

Hence

$$t \leq \frac{4(a_1 + a_2)}{\epsilon} \leq \frac{3072 N(\epsilon)}{\epsilon^2}.$$

In summary, we have

$$t \leq \frac{4(a_1 + a_2)}{\epsilon} \leq \frac{3072 N(\epsilon)}{\epsilon^2},$$

and hence

$$\mathrm{Vol}(-K_X) \leq \frac{24 \cdot 3072^2 N(\epsilon)^2}{\epsilon^4}.$$

We have completed the proof of Theorem 1.1.12.

3.4 Boundedness of log Fano threefolds of fixed index

In this section, we prove the boundedness of log Fano threefolds of fixed index (Corollary 1.1.8). Corollary 1.1.8 follows directly by Theorem 1.1.7 and the following more general theorem which might be well known to experts.

Theorem 3.4.1. *Fix positive integers r and n. Assume Weak BAB Conjecture holds in dimension n.*

Let $\mathcal{D}$ be the set of all normal projective varieties X, where $\dim X = n$, K_X is $\mathbb{Q}$-Cartier, and there exists an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is klt and $-r(K_X + \Delta)$ is Cartier and ample.

Then $\mathcal{D}$ forms a bounded family.

Proof. Consider a klt log Fano pair (X, Δ) of dimension n such that K_X is $\mathbb{Q}$-Cartier and $-r(K_X + \Delta)$ is Cartier.

Note that (X, Δ) is ϵ-klt with $\epsilon = 1/2r$ by the assumption. It follows that $(-(K_X + \Delta))^n \leq M(n, \epsilon)$ by Weak BAB Conjecture in dimension n.

Since $-r(K_X+\Delta)$ is Cartier and ample, by Kollár's Effective Base Point Free Theorem [Kol93, 1.1 Theorem, 1.2 Lemma], $G := -Nr(K_X+\Delta)$ is very ample for $N = 2n \cdot (n+3)!$.

Note that $(K_X + G) \cdot C \geq 0$ for all curves C satisfying $K_X \cdot C \geq -2n$. Hence by Cone Theorem (see [KoMo98, Theorem 3.7]), $K_X + G$ is nef.

Now we can bound G^n and $|-K_X \cdot G^{n-1}|$ from above. Clearly $G^n \leq N^n r^n M(n, \epsilon)$ by definition and $-K_X \cdot G^{n-1} > 0$ since $-K_X$ is big. On the other hand,

$$\begin{aligned} & -K_X \cdot G^{n-1} \\ = & -(K_X+G) \cdot G^{n-1} + G \cdot G^{n-1} \\ \leq & \, N^n r^n M(n, \epsilon). \end{aligned}$$

Hence G^n and $|-K_X \cdot G^{n-1}|$ are bounded from above. By [KoMa83], the coefficients of the Hilbert polynomial $P(t) = \chi(X, \mathcal{O}(tG))$ is bounded and hence there are only finitely many Hilbert polynomials for the polarized variety (X, G). And hence X is in a bounded family.

We complete the proof. $\square$

4 On the anti-canonical geometry of $\mathbb{Q}$-Fano threefolds

In this chapter, we investigate (weak) $\mathbb{Q}$-Fano 3-folds. We will prove Theorems 1.2.9, 1.2.11, 1.2.13, 1.2.14, and 1.2.15.

This chapter is organized as follows. In Section 4.1, we give an explicit bound of Gorenstein indices of weak $\mathbb{Q}$-Fano 3-folds. In Section 4.2, we consider Problem 1.2.8 on $\mathbb{Q}$-Fano 3-folds. We generalize a result of Alexeev and reduce the problem to the numerical behavior of anti-plurigenera, then we apply a method developed by J. A. Chen and M. Chen to analyze the possible weighted baskets. Section 4.3 is devoted to proving Theorem 1.2.14 for weak $\mathbb{Q}$-Fano 3-folds. We reduce the problem to the numerical behavior of Hilbert polynomials and use Reid's formula to estimate the lower bound of Hilbert polynomials. Finally we study the birationality and generic finiteness in Section 4.4. We give an effective criterion for the birationality (resp. generic finiteness) of φ_{-m}. As applications, we prove Theorems 1.2.11 and 1.2.15.

4.1 Upper bound of Gorenstein indices

The following fact might be known to experts. We will apply it in our argument.

Proposition 4.1.1. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Then either $r_X = 840$ or $r_X \leq 660$.* [1]

[1]This means that the Gorenstein index of a weak $\mathbb{Q}$-Fano 3-fold is bounded from above by 840. Among known $\mathbb{Q}$-Fano 3-folds, the maximal Gorenstein index is 420. For example, so is the general weighted hypersurface $X_{19} \subset \mathbb{P}(1,3,4,5,7)$ (cf. [IF00, List 16.6, No.40]). We do not know if this bound is optimal.

Proof. Write Reid's basket

$$B_X = \{(b_i, r_i) \mid i = 1, \cdots, s; 0 < b_i \le \frac{r_i}{2}; b_i \text{ is coprime to } r_i\}.$$

Then, by definition, $r_X = \text{l.c.m.}\{r_i \mid i = 1, \cdots, s\}$.

By [KMMT00], we know that $(-K_X \cdot c_2(X)) \ge 0$. Therefore Reid [Reid87, 10.3] gives the inequality

$$\sum_i (r_i - \frac{1}{r_i}) \le 24. \tag{4.1.1}$$

Now for the sequence $\mathcal{R} = (r_i)_i$, we define a new set $\mathcal{P} = \{s_j\}_j$ as following: if we factor r_i by primes such that $r_i = p_1^{a_{1i}} p_2^{a_{2i}} \cdots p_k^{a_{ki}}$, then we take $\mathcal{P} = \{p_j^{a_{ji}}\}_{1 \le j \le k, i}$. Roughly speaking, we split r_i by its prime factors. It is easy to show that if $a, b > 1$ and coprime, then

$$ab - \frac{1}{ab} \ge a - \frac{1}{a} + b - \frac{1}{b} + 2. \tag{4.1.2}$$

So

$$\sum_j (s_j - \frac{1}{s_j}) \le \sum_i (r_i - \frac{1}{r_i}) \le 24. \tag{4.1.3}$$

And we also have $\text{l.c.m.}(s_j)_j = \text{l.c.m.}(r_i)_i = r_X$. So the problem is reduced to treat the sequence $(s_j)_j$ instead. Clearly, for each i,

$$s_i \in \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19\}.$$

Now we may assume that $r_X > 660$.

Denote by s_1 the largest value in $\mathcal{P}$, by s_2 the second largest value, by s_3, s_4 the third, the forth, and so on. For instance, if $\mathcal{P} = \{2, 3, 4, 5\}$, then $s_1 = 5$, $s_2 = 4$, $s_3 = 3$, and $s_4 = 2$. If the value s_j does not exist by definition, then we set $s_j = 1$. In the previous example, we have $s_5 = 1$.

Since $\text{l.c.m.}(2, 3, 4, 5, 7) = 420$ and $\text{l.c.m.}(2, 3, 4, 5, 7, 8) = 840$, if $s_1 \le 8$, then $3, 5, 7, 8 \in \mathcal{P}$. In this case $\mathcal{P} = \{3, 5, 7, 8\}$ or $\{2, 3, 5, 7, 8\}$ by inequality (4.1.3) and $\mathcal{R} = (3, 5, 7, 8)$ or $(2, 3, 5, 7, 8)$ by inequality (4.1.2). In a word, $r_X = 840$.

If $s_1 \ge 16$, then

$$\sum_{j>1} (s_j - \frac{1}{s_j}) \le 8 + \frac{1}{16}.$$

Then $s_2 \leq 8$. Also $s_2 \geq 5$ since, otherwise, $\text{l.c.m.}(2, 3, 4, s_1) \leq 228 < r_X$ (a contradiction). Hence

$$\sum_{j>2}(s_j - \frac{1}{s_j}) \leq 3 + \frac{1}{16} + \frac{1}{5}.$$

So $s_3 \leq 3$, but 2 and 3 can not be in $\mathcal{P}$ simultaneously. Then $\text{l.c.m.}(s_j)_j \leq 3 \times 8 \times 19 < r_X$, a contradiction.

If $s_1 = 13$, then $s_2 \geq 5$ since, otherwise, $\text{l.c.m.}(2, 3, 4, s_1) = 12s_1 < r_X$ (a contradiction). Then

$$\sum_{j>2}(s_j - \frac{1}{s_j}) \leq 11 - s_2 + \frac{1}{13} + \frac{1}{s_2}.$$

If $s_2 = 11$, then $s_j = 1$ for any $j > 2$ and $r_X = 11 \times 13$, a contradiction. If $s_2 = 9$, then $s_3 \leq 2$ and $\text{l.c.m.}(s_j)_j \leq 2 \times 9 \times 13 < r_X$, a contradiction. If $s_2 = 8$, then $s_3 \leq 3$, but 2 and 3 can not be in $\mathcal{P}$ simultaneously. So $\text{l.c.m.}(s_j)_j \leq 3 \times 8 \times 13 < r_X$, a contradiction. If $s_2 = 7$, then $s_3 \leq 4$, but 3 and 4 can not be in $\mathcal{P}$ simultaneously. So $\text{l.c.m.}(s_j)_j \leq 6 \times 7 \times 13 < r_X$, a contradiction. If $s_2 = 5$, then 3 and 4 can not be in $\mathcal{P}$ simultaneously. So $\text{l.c.m.}(s_j)_j \leq 6 \times 5 \times 13 < r_X$, a contradiction.

If $s_1 = 11$, then $9 \geq s_2 \geq 7$ since, otherwise, $\text{l.c.m.}(2, 3, 4, 5, s_1) = 60s_1 < r_X$ (a contradiction). Then

$$\sum_{j>2}(s_j - \frac{1}{s_j}) \leq 6 + \frac{1}{11} + \frac{1}{7}.$$

Hence $s_3 \leq 5$. If $s_3 = 5$, then $s_j = 1$ for any $j > 3$ and $\text{l.c.m.}(s_j)_j \leq 5 \times 9 \times 11 < r_X$, a contradiction. If $s_3 = 4$, then $s_4 \leq 2$ and $\text{l.c.m.}(s_j)_j \leq 4 \times 9 \times 11 < r_X$, a contradiction. If $s_3 \leq 3$, then $\text{l.c.m.}(s_j)_j \leq 2 \times 3 \times 9 \times 11 < r_X$, a contradiction.

If $s_1 = 9$, then $8 \geq s_2 \geq 7$ since, otherwise, $\text{l.c.m.}(2, 3, 4, 5, 9) = 180 < r_X$ (a contradiction). Consider firstly the case $s_2 = 8$. We have

$$\sum_{j>2}(s_j - \frac{1}{s_j}) \leq 7 + \frac{1}{9} + \frac{1}{8}.$$

If $s_3 = 7$, then $s_j = 1$ for any $j > 3$ and $\text{l.c.m.}(s_j)_j \leq 7 \times 8 \times 9 < r_X$, a contradiction. If $s_3 \leq 5$, then $\text{l.c.m.}(s_j)_j \leq \text{lcm}(2, 3, 4, 5, 8, 9) = 360 < r_X$, a contradiction. Next we consider the case $s_2 = 7$. Then

$$\sum_{j>2}(s_j - \frac{1}{s_j}) \leq 8 + \frac{1}{9} + \frac{1}{7}.$$

If $s_3 = 5$, then $s_4 \leq 3$ and l.c.m.$(s_j)_j \leq 2 \times 5 \times 7 \times 9 < r_X$, a contradiction. If $s_3 \leq 4$, then l.c.m.$(s_j)_j \leq 4 \times 7 \times 9 < r_X$, a contradiction. So we conclude the statement.

From the proof we also know that $r_X = 840$ only happens when $\mathcal{R} = (3, 5, 7, 8)$ or $(2, 3, 5, 7, 8)$. $\square$

4.2 When is $|-mK_X|$ not composed with a pencil (Part I)?

The most important part of this chapter is to find a minimal positive integer m so that $|-mK_X|$ is not composed with a pencil of surfaces. For the convenience of expression, we fix our notations first.

Definition 4.2.1. Let X be a weak $\mathbb{Q}$-Fano 3-fold. For any $0 \leq i \leq 2$, define

$$\delta_i(X) := \min\{m \in \mathbb{Z}^+ \mid \dim \overline{\varphi_{-m}(X)} > i\}.$$

We will mainly treat $\mathbb{Q}$-Fano 3-folds in this section.

4.2.1 Two key theorems

We prove two theorems here which are crucial in proving Theorem 1.2.9.

Theorem 4.2.2. *Let X be a $\mathbb{Q}$-Fano 3-fold with the basket B of singularities. Fix a positive integer m such that $P_{-m} > 0$. Assume that, for each pair $(b, r) \in B$, one of the following conditions is satisfied:*

(1) $m \equiv 0, \pm 1 \mod r$*;*

(2) $m \equiv -2 \mod r$ *and* $b = \lfloor \frac{r}{2} \rfloor$*;*

(3) $m \equiv 2 \mod r$ *and* $3b \geq r$*;*

(4) $m \equiv 3 \mod r$ *and* $4b \geq r$*;*

(5) $m \equiv 4 \mod r$, $\overline{b}(r - \overline{b}) \geq \overline{4b}(r - \overline{4b})$, *and*

$$\overline{b}(r - \overline{b}) + \overline{2b}(r - \overline{2b}) \geq \overline{3b}(r - \overline{3b}) + \overline{4b}(r - \overline{4b}).$$

Then one of the following holds:

(I) $P_{-m} = 1$ *and* $-mK_X \sim E$ *is a fixed prime divisor;*

(II) $P_{-m} = 2$, $|-mK_X|$ *does not have fixed part and is composed with an irreducible rational pencil of surfaces;*

(III) $P_{-m} \geq 3$, $|-mK_X|$ *does not have fixed part and is not composed with a pencil of surfaces.*

Proof. We generalize the argument of Alexeev [Ale94b, 2.18] where the case $m = 1$ is treated.

Assume that none of the conclusions holds, then there exists a strictly effective divisor E such that $-mK_X - E$ is strictly effective and

$$h^0(-mK_X) - h^0(-mK_X - E) - h^0(E) + h^0(\mathcal{O}_X) = 0.$$

In fact, if $P_{-m} = 1$ and $-mK_X \sim D$ is not a prime divisor, then we take E to be one irreducible component of D; if $P_{-m} \geq 2$ and $|-mK_X|$ has fixed part, then we take E to be one component in the fixed part; if $P_{-m} \geq 3$, $|-mK_X|$ does not have fixed part and is composed with a (rational) pencil of surfaces, then $|-mK_X| = |nS|$ with $n \geq 2$ and we can take $E = S$.

By Kawamata–Viehweg vanishing theorem and $\rho(X) = 1$, all higher cohomologies vanish for $\mathcal{O}_X(-mK_X)$, $\mathcal{O}_X(-mK_X - E)$, $\mathcal{O}_X(E)$, and $\mathcal{O}_X$. Hence

$$\Delta\Delta_{\chi}(-mK_X, -mK_X - E, E, 0) = 0,$$

where the *double difference* of a function f is defined by

$$\Delta\Delta_f(a, a-d, b, b-d) = f(a) - f(a-d) - f(b) + f(b-d).$$

Then we have

$$\Delta\Delta_{\chi,\mathrm{reg}}(-mK_X, -mK_X - E, E, 0) + \Delta\Delta_{\chi,\mathrm{sing}}(-mK_X, -mK_X - E, E, 0) = 0.$$

It is clear to see that

$$\Delta\Delta_{\chi,\mathrm{reg}}(-mK_X, -mK_X - E, E, 0) = \frac{m+1}{2}(-K_X)(-mK_X - E)E > 0,$$

since E and $-mK_X - E$ are ample by our construction and $\rho(X) = 1$. To get a contradiction, it is sufficient to show that

$$\Delta\Delta_{\chi,\mathrm{sing}}(-mK_X, -mK_X - E, E, 0) \geq 0$$

under the assumption of this theorem. Thus it amounts to show that, for every single point $Q = (b, r) \in B$,

$$c_Q(-mK_X) - c_Q(-mK_X - E) - c_Q(E) \geq 0. \tag{4.2.1}$$

Set $F(x) := \frac{\overline{x}(r-\overline{x})}{2r}$ for any integer x and $l := \overline{m}$. We may assume that the local index of E at Q is i $(0 \leq i < r)$.

Then

$$
\begin{aligned}
& c_Q(-mK_X) - c_Q(-mK_X - E) - c_Q(E) \\
= & \Big(- \frac{(2r-l)(r^2-1)}{12r} + \sum_{j=0}^{2r-l-1} F(jb) \Big) \\
& - \Big(- \frac{(2r-l-i)(r^2-1)}{12r} + \sum_{j=0}^{2r-l-i-1} F(jb) \Big) \\
& - \Big(- \frac{i(r^2-1)}{12r} + \sum_{j=0}^{i-1} F(jb) \Big) \\
= & \sum_{j=0}^{2r-l-1} F(jb) - \sum_{j=0}^{2r-l-i-1} F(jb) - \sum_{j=0}^{i-1} F(jb) \\
= & \sum_{j=2r-l-i}^{2r-l-1} F(jb) - \sum_{j=0}^{i-1} F(jb) \\
= & \sum_{j=l+1}^{l+i} F(jb) - \sum_{j=0}^{i-1} F(jb) \\
= & \sum_{j=i}^{l+i} F(jb) - \sum_{j=0}^{l} F(jb) \\
= & \sum_{j=0}^{l} F(ib+jb) - \sum_{j=0}^{l} F(jb). \qquad (4.2.2)
\end{aligned}
$$

Then to prove inequality (4.2.1), it suffices to prove that

$$G(x) := \sum_{j=0}^{l} F(x+jb) - \sum_{j=0}^{l} F(jb) \geq 0$$

for arbitrary integer x.

Note that $G(x)$ is a periodic piecewisely quadratic function with negative leading coefficients. Hence the minimal value can only be reached at end points of each piece. It is easy to see that the set of end points is $\{nr - jb \mid n \in \mathbb{Z}, j = 0, 1, \ldots, l\}$. Hence $G(x) \geq 0$ is equivalent to $G(-jb) \geq 0$ for all $j = 0, 1, \ldots, l$. Note that $G(0) = G(-lb) = 0$.

If $m \equiv 0, 1 \mod r$, there is nothing to prove.

If $m \equiv 2 \mod r$, then $G(-b) = F(b) - F(2b)$. It is easy to see that $F(b) - F(2b) \geq 0$ is equivalent to $3b \geq r$.

If $m \equiv 3 \mod r$, then $G(-b) = G(-2b) = F(b) - F(3b)$. It is easy to see that $F(b) - F(3b) \geq 0$ is equivalent to $4b \geq r$.

If $m \equiv 4 \mod r$, then $G(-b) = G(-3b) = F(b) - F(4b)$ and $G(-2b) = F(b) + F(2b) - F(3b) - F(4b)$.

If $m \equiv -1 \mod r$, then $G(x) = \sum_{j=0}^{r-1} F(x + jb) - \sum_{j=0}^{r-1} F(jb) = 0$.

If $m \equiv -2 \mod r$, then $G(x) = \sum_{j=0}^{r-2} F(x + jb) - \sum_{j=0}^{r-2} F(jb) = F(b) - F(x + (r-1)b)$. And $F(b) - F(x + (r-1)b) \geq 0$ for all x if and only if $b = \lfloor \frac{r}{2} \rfloor$.

So we have proved the theorem. □

As a corollary of Theorem 4.2.2, we know the geometry of $|-K_X|$ when P_{-1} is large due to Alexeev.

Corollary 4.2.3 ([Ale94b, 2.18])**.** *Let X be a $\mathbb{Q}$-Fano 3-fold. If $P_{-1} \geq 3$, then $|-K_X|$ has no fixed part and is not composed with a pencil of surfaces.*

Hence we only need to deal with the case when $P_{-1} < 3$. For this purpose, we prove the following theorem.

Theorem 4.2.4. *Let X be a $\mathbb{Q}$-Fano 3-fold. Fix a positive integer m. Assume that one of the following holds:*

(i) $P_{-m} = 1$ and $E \in |-mK_X|$ is a fixed prime divisor;

(ii) $P_{-m} = 2$ and $|-mK_X|$ does not have fixed part.

Write $n_0 := \min\{n \in \mathbb{Z}^+ \mid P_{-nm} \geq 2\}$. For any integer $l \geq n_0$, write $l = sn_0 + t$ with $s \in \mathbb{Z}$ and $0 \leq t \leq n_0 - 1$. Take $l_0 = \min\{l \in \mathbb{Z}_{\geq n_0} \mid P_{-lm} > s + 1\}$. Then $|-l_0 mK_X|$ does not have fixed part and is not composed with a pencil of surfaces.

Proof. First we assume that $|-l_0 mK_X|$ has a base component E_{l_0}. It follows that $P_{-m} = 1$ and $E_{l_0} = E$. Thus, by definition, we have $l_0 > 1$. Hence

$$\begin{aligned} P_{-(l_0-1)m} &= h^0(-l_0 mK_X - (-mK_X)) \\ &= h^0(-l_0 mK_X - E_{l_0}) = h^0(-l_0 mK_X) > s + 1, \end{aligned}$$

which contradicts the minimality of l_0. The similar argument implies that $|-n_0 mK_X|$ does not have fixed part.

Now assume that $|-l_0 mK_X|$ is composed with a (rational) pencil of surfaces, i.e.

$$|-l_0 mK_X| = |(P_{-l_0 m} - 1)S|,$$

where $|S|$ is an irreducible rational pencil. Write $l_0 = sn_0 + t$. Since $P_{-n_0m} \geq 2$, we have $P_{-sn_0m} \geq s+1$.

If $t > 0$, by the minimality of l_0 we get $P_{-sn_0m} = s+1$. So we can write $|-sn_0mK_X| = |sS|$ by Lemma 2.4.2 since $|-n_0mK_X|$ does not have fixed part and $|-sn_0mK_X| \preceq |-l_0mK_X|$. Now

$$\begin{aligned}-tmK_X &\sim -l_0mK_X - (-sn_0mK_X) \sim (P_{-l_0m} - 1)S - sS \\ &= (P_{-l_0m} - 1 - s)S \geq S.\end{aligned}$$

This implies that $P_{-tm} \geq 2$, which contradicts the minimality of n_0. Hence $t = 0$ and $l_0 = sn_0$.

If $s \geq 2$, by the minimality of l_0 we get $P_{-(s-1)n_0m} = s \geq 2$. We can write $|-(s-1)n_0mK_X| = |(s-1)S|$ by Lemma 2.4.2. Hence

$$\begin{aligned}-n_0mK_X &\sim -l_0mK_X - (-(s-1)n_0mK_X) \sim (P_{-l_0m} - 1)S - (s-1)S \\ &= (P_{-l_0m} - s)S \geq 2S.\end{aligned}$$

This implies that $P_{-n_0m} \geq 3$, which contradicts the minimality of l_0.

Hence $s = 1$ and $l_0 = n_0$. By $P_{-n_0m} \geq 3$, we have $n_0 > 1$. This implies, by assumption, $P_{-m} = 1$ and $-mK_X \sim E$ is a fixed prime divisor. Since $E \leq (P_{-Nm} - 1)S \sim -n_0mK_X$ and E is reduced and irreducible, $E \leq S_0$ for certain surface $S_0 \in |S|$. Hence

$$\begin{aligned}-(n_0 - 1)mK_X &\sim -n_0mK_X - (-mK_X) \sim (P_{-n_0m} - 1)S - E \\ &\geq (P_{-n_0m} - 2)S + (S_0 - E) \geq S.\end{aligned}$$

This implies that $P_{-(n_0-1)m} \geq 2$, which contradicts the minimality of n_0. We are done. $\square$

Now let us explain the strategy to prove Theorem 1.2.9. Firstly, we divide all $\mathbb{Q}$-Fano 3-folds into several families, roughly speaking, by the value of P_{-1}. Then in each family, we may take a proper m satisfying the condition of Theorem 4.2.2. Applying Theorem 4.2.4 to m, we are able to find the number l_0 and so $\delta_1(X) \leq l_0m$. In order to find such l_0, or an upper bound of l_0, we may assume that l_0 is sufficiently large, say, $l_0 \geq 9$, then by the assumption of Theorem 4.2.4, we know the value of $P_{-m}, P_{-2m}, P_{-3m}, \ldots, P_{-8m}$. Then, by Chen–Chen's method ([CC08]) on the analysis of baskets, we can recover all possibilities for baskets of singularities, of which each possibility can be proved to be either impossible or very easy to treat. For this purpose, we need to recall relevant materials on baskets, packings, the canonical sequence and so on.

4.2.2 Weighted baskets

All contents of this subsection are mainly from Chen–Chen [CC08, CC10a]. We list them as follows:

1. Let $B = \{(b_i, r_i) \mid i = 1, \cdots, s; 0 < b_i \leq \frac{r_i}{2}; b_i \text{ is coprime to } r_i\}$ be a basket. We set $\sigma(B) := \sum_i b_i$, $\sigma'(B) := \sum_i \frac{b_i^2}{r_i}$, and $\Delta^n(B) = \sum_i (\frac{\overline{b_i n}(r_i - \overline{b_i n})}{2r_i} - \frac{b_i n(r_i - b_i n)}{2r_i})$ for any integer $n > 1$.

2. The new (generalized) basket
$$B' := \{(b_1 + b_2, r_1 + r_2), (b_3, r_3), \cdots, (b_s, r_s)\}$$
is called a *packing* of B, denoted as $B \succeq B'$. Note that $\{(2,4)\} = \{(1,2),(1,2)\}$. We call $B \succ B'$ a *prime packing* if $b_1 r_2 - b_2 r_1 = 1$. A composition of finite packings is also called a packing. So the relation "$\succeq$" is a partial ordering on the set of baskets.

3. Note that for a weak $\mathbb{Q}$-Fano 3-fold X, all the anti-plurigenera P_{-n} can be determined by Reid's basket B_X and $P_{-1}(X)$. This leads to the notion of "weighted basket". We call a pair $\mathbb{B} = (B, \tilde{P}_{-1})$ a *weighted basket* if B is a basket and $\tilde{P}_{-1}$ is a non-negative integer. We write $(B, \tilde{P}_{-1}) \succeq (B', \tilde{P}_{-1})$ if $B \succeq B'$.

4. Given a weighted basket $\mathbb{B} = (B, \tilde{P}_{-1})$, define $\tilde{P}_{-1}(\mathbb{B}) := \tilde{P}_{-1}$ and the volume
$$-K^3(\mathbb{B}) := 2\tilde{P}_{-1} + \sigma(B) - \sigma'(B) - 6.$$
For all $m \geq 1$, we define the "anti-plurigenus" in the following inductive way:
$$\begin{aligned} &\tilde{P}_{-(m+1)} - \tilde{P}_{-m} \\ = &\frac{1}{2}(m+1)^2(-K^3(\mathbb{B}) + \sigma'(B)) + 2 - \frac{m+1}{2}\sigma - \Delta^{m+1}(B). \end{aligned}$$
Note that, if we set $\mathbb{B} = (B_X, P_{-1}(X))$ for a given weak $\mathbb{Q}$-Fano 3-fold X, then we can verify directly that $-K^3(\mathbb{B}) = -K_X^3$ and $\tilde{P}_{-m}(\mathbb{B}) = P_{-m}(X)$ for all $m \geq 1$.

Property 4.2.5 ([CC10a, Section 3]). *Assume* $\mathbb{B} := (B, \tilde{P}_{-1}) \succeq \mathbb{B}' := (B', \tilde{P}_{-1})$. *Then*

(i) $\sigma(B) = \sigma(B')$ *and* $\sigma'(B) \geq \sigma'(B')$;

(ii) *For all integer* $n \geq 1$, $\Delta^n(B) \geq \Delta^n(B')$;

(iii) $-K^3(\mathbb{B}) + \sigma'(B) = -K^3(\mathbb{B}') + \sigma'(B')$*;*

(iv) $-K^3(\mathbb{B}) \le -K^3(\mathbb{B}')$*;*

(v) $\tilde{P}_{-m}(\mathbb{B}) \le \tilde{P}_{-m}(\mathbb{B}')$ *for all* $m \ge 2$*.*

Next we recall the "canonical" sequence of a basket B. Set $S^{(0)} := \{\frac{1}{n} \mid n \ge 2\}$, $S^{(5)} := S^{(0)} \cup \{\frac{2}{5}\}$, and inductively for all $n \ge 5$,

$$S^{(n)} := S^{(n-1)} \cup \{\frac{b}{n} \mid 0 < b < \frac{n}{2},\ b \text{ is coprime to } n\}.$$

Each set $S^{(n)}$ gives a division of the interval $(0, \frac{1}{2}] = \bigcup_i [\omega_{i+1}^{(n)}, \omega_i^{(n)}]$ with $\omega_i^{(n)}, \omega_{i+1}^{(n)} \in S^{(n)}$. Let $\omega_{i+1}^{(n)} = \frac{q_{i+1}}{p_{i+1}}$ and $\omega_i^{(n)} = \frac{q_i}{p_i}$ with g.c.d$(q_l, p_l) = 1$ for $l = i, i+1$. Then it is easy to see that $q_i p_{i+1} - p_i q_{i+1} = 1$ for all n and i (cf. [CC10a, Claim A]).

Now given a basket $B = \{(b_i, r_i) \mid i = 1, \cdots, s\}$, we define new baskets $\mathscr{B}^{(n)}(B)$, where $\mathscr{B}^{(n)}(\cdot)$ can be regarded as an operator on the set of baskets. For each $(b_i, r_i) \in B$, if $\frac{b_i}{r_i} \in S^{(n)}$, then we set $\mathscr{B}_i^{(n)} := \{(b_i, r_i)\}$. If $\frac{b_i}{r_i} \notin S^{(n)}$, then $\omega_{l+1}^{(n)} < \frac{b_i}{r_i} < \omega_l^{(n)}$ for some l. We write $\omega_l^{(n)} = \frac{q_l}{p_l}$ and $\omega_{l+1}^{(n)} = \frac{q_{l+1}}{p_{l+1}}$ respectively. In this situation, we can unpack (b_i, r_i) to $\mathscr{B}_i^{(n)} := \{(r_i q_l - b_i p_l) \times (q_{l+1}, p_{l+1}), (-r_i q_{l+1} + b_i p_{l+1}) \times (q_l, p_l)\}$. Adding up those $\mathscr{B}_i^{(n)}$, we get a new basket $\mathscr{B}^{(n)}(B)$, which is uniquely defined according to the construction and $\mathscr{B}^{(n)}(B) \succeq B$ for all n. Note that, by our definition, $B = \mathscr{B}^{(n)}(B)$ for sufficiently large n.

Moreover, we have

$$\mathscr{B}^{(n-1)}(B) = \mathscr{B}^{(n-1)}(\mathscr{B}^{(n)}(B)) \succeq \mathscr{B}^{(n)}(B)$$

for all $n \ge 1$ (cf. [CC10a, Claim B]). Therefore we have a chain of baskets

$$\mathscr{B}^{(0)}(B) \succeq \mathscr{B}^{(5)}(B) \succeq \cdots \succeq \mathscr{B}^{(n)}(B) \succeq \cdots \succeq B.$$

The step $\mathscr{B}^{(n-1)}(B) \succeq \mathscr{B}^{(n)}(B)$ can be achieved by a number of successive prime packings. Let $\epsilon_n(B)$ be the number of such prime packings. For any $n > 0$, set $B^{(n)} := \mathscr{B}^{(n)}(B)$.

The following properties are essential to represent $\mathscr{B}^{(n)}(B)$.

Lemma 4.2.6 ([CC10a, Lemma 2.16])**.** *For the above sequence* $\{\mathscr{B}^{(n)}(B)\}$*, the following statements hold:*

(i) $\Delta^j(\mathscr{B}^{(0)}(B)) = \Delta^j(B)$ *for* $j = 3, 4$*;*

(ii) $\Delta^j(\mathcal{B}^{(n-1)}(B)) = \Delta^j(\mathcal{B}^{(n)}(B))$ *for all* $j < n$*;*

(iii) $\Delta^n(\mathcal{B}^{(n-1)}(B)) = \Delta^n(\mathcal{B}^{(n)}(B)) + \epsilon_n(B)$.

It follows that $\Delta^j(\mathcal{B}^{(n)}(B)) = \Delta^j(B)$ for all $j \le n$ and

$$\epsilon_n(B) = \Delta^n(\mathcal{B}^{(n-1)}(B)) - \Delta^n(\mathcal{B}^{(n)}(B)) = \Delta^n(\mathcal{B}^{(n-1)}(B)) - \Delta^n(B).$$

Moreover, given a weighted basket $\mathbb{B} = (B, \tilde{P}_{-1})$, we can similarly consider $\mathcal{B}^{(n)}(\mathbb{B}) := (\mathcal{B}^{(n)}(B), \tilde{P}_{-1})$. It follows that

$$\tilde{P}_{-j}(\mathcal{B}^{(n)}(\mathbb{B})) = \tilde{P}_{-j}(\mathbb{B}) \text{ for all } j \le n.$$

Therefore we can realize the canonical sequence of weighted baskets as an approximation of weighted baskets via anti-plurigenera.

We now recall the relation between weighted baskets and anti-plurigenera more closely. For a given weighted basket $\mathbb{B} = (B, \tilde{P}_{-1})$, we start by computing the non-negative number ϵ_n and $B^{(0)}$, $B^{(5)}$ in terms of $\tilde{P}_{-m}$. From the definition of $\tilde{P}_{-m}$ we get

$$\begin{aligned}
\sigma(B) &= 10 - 5\tilde{P}_{-1} + \tilde{P}_{-2},\\
\Delta^{m+1} &= (2 - 5(m+1) + 2(m+1)^2) + \frac{1}{2}(m+1)(2-3m)\tilde{P}_{-1}\\
&\quad + \frac{1}{2}m(m+1)\tilde{P}_{-2} + \tilde{P}_{-m} - \tilde{P}_{-(m+1)}.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\Delta^3 &= 5 - 6\tilde{P}_{-1} + 4\tilde{P}_{-2} - \tilde{P}_{-3};\\
\Delta^4 &= 14 - 14\tilde{P}_{-1} + 6\tilde{P}_{-2} + \tilde{P}_{-3} - \tilde{P}_{-4}.
\end{aligned}$$

Assume $B^{(0)} = \{n^0_{1,r} \times (1, r) \mid r \ge 2\}$. By Lemma 4.2.6, we have

$$\begin{aligned}
\sigma(B) = \sigma(B^{(0)}) &= \sum n^0_{1,r};\\
\Delta^3(B) = \Delta^3(B^{(0)}) &= n^0_{1,2};\\
\Delta^4(B) = \Delta^4(B^{(0)}) &= 2n^0_{1,2} + n^0_{1,3}.
\end{aligned}$$

Thus we get $B^{(0)}$ as follows:

$$\begin{cases}
n^0_{1,2} = 5 - 6\tilde{P}_{-1} + 4\tilde{P}_{-2} - \tilde{P}_{-3};\\
n^0_{1,3} = 4 - 2\tilde{P}_{-1} - 2\tilde{P}_{-2} + 3\tilde{P}_{-3} - \tilde{P}_{-4};\\
n^0_{1,4} = 1 + 3\tilde{P}_{-1} - \tilde{P}_{-2} - 2\tilde{P}_{-3} + \tilde{P}_{-4} - \sigma_5;\\
n^0_{1,r} = n^0_{1,r}, r \ge 5,
\end{cases}$$

where $\sigma_5 := \sum_{r\geq 5} n^0_{1,r}$. A computation gives

$$\epsilon_5 = 2 + \tilde{P}_{-2} - 2\tilde{P}_{-4} + \tilde{P}_{-5} - \sigma_5.$$

Therefore we get $B^{(5)}$ as follows:

$$\begin{cases} n^5_{1,2} = 3 - 6\tilde{P}_{-1} + 3\tilde{P}_{-2} - \tilde{P}_{-3} + 2\tilde{P}_{-4} - \tilde{P}_{-5} + \sigma_5; \\ n^5_{2,5} = 2 + \tilde{P}_{-2} - 2\tilde{P}_{-4} + \tilde{P}_{-5} - \sigma_5; \\ n^5_{1,3} = 2 - 2\tilde{P}_{-1} - 3\tilde{P}_{-2} + 3\tilde{P}_{-3} + \tilde{P}_{-4} - \tilde{P}_{-5} + \sigma_5; \\ n^5_{1,4} = 1 + 3\tilde{P}_{-1} - \tilde{P}_{-2} - 2\tilde{P}_{-3} + \tilde{P}_{-4} - \sigma_5; \\ n^5_{1,r} = n^0_{1,r}, r \geq 5. \end{cases}$$

Because $B^{(5)} = B^{(6)}$, we see $\epsilon_6 = 0$ and on the other hand

$$\epsilon_6 = 3\tilde{P}_{-1} + \tilde{P}_{-2} - \tilde{P}_{-3} - \tilde{P}_{-4} - \tilde{P}_{-5} + \tilde{P}_{-6} - \epsilon = 0$$

where $\epsilon := 2\sigma_5 - n^0_{1,5} \geq 0$.

Going on a similar calculation, we get

$$\begin{aligned} \epsilon_7 &= 1 + \tilde{P}_{-1} + \tilde{P}_{-2} - \tilde{P}_{-5} - \tilde{P}_{-6} + \tilde{P}_{-7} - 2\sigma_5 + 2n^0_{1,5} + n^0_{1,6}; \\ \epsilon_8 &= 2\tilde{P}_{-1} + \tilde{P}_{-2} + \tilde{P}_{-3} - \tilde{P}_{-4} - \tilde{P}_{-5} - \tilde{P}_{-7} + \tilde{P}_{-8} \\ &\quad - 3\sigma_5 + 3n^0_{1,5} + 2n^0_{1,6} + n^0_{1,7}. \end{aligned}$$

A weighted basket $\mathbb{B} = (B, \tilde{P}_{-1})$ is said to be *geometric* if $\mathbb{B} = (B_X, P_{-1}(X))$ for a $\mathbb{Q}$-Fano 3-fold X. Geometric baskets are subject to some geometric properties. By [Kaw92a], we have that $(-K_X \cdot c_2(X)) > 0$. Therefore [Reid87, 10.3] gives the inequality

$$\gamma(B) := \sum_i \frac{1}{r_i} - \sum_i r_i + 24 > 0. \tag{4.2.3}$$

For packings, it is easy to see the following lemma.

Lemma 4.2.7. *Given a packing of baskets $B_1 \succeq B_2$, we have $\gamma(B_1) \geq \gamma(B_2)$. In particular, if inequality (4.2.3) does not hold for B_1, then it does not hold for B_2.*

Lemma 4.2.7 implies that, for two weighted baskets $\mathbb{B}_1 \succeq \mathbb{B}_2$, if $\mathbb{B}_1$ is non-geometric, then neither is $\mathbb{B}_2$.

Furthermore, $-K^3(\mathbb{B}) = -K^3_X > 0$ gives the inequality

$$\sigma'(B) < 2P_{-1} + \sigma(B) - 6. \tag{4.2.4}$$

Finally, by [Kol95, Lemma 15.6.2], if $P_{-m} > 0$ and $P_{-n} > 0$, then

$$P_{-m-n} \geq P_{-m} + P_{-n} - 1. \tag{4.2.5}$$

4.2.3 $\mathbb{Q}$-Fano 3-folds with $h^0(-K) = 2$

In this subsection we prove the following theorem.

Theorem 4.2.8. *Let X be a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 2$. Then for any integer $m \geq 6$, $\dim \overline{\varphi_{-m}(X)} > 1$. In particular, $\delta_1(X) \leq 6$.*

Theorem 4.2.8 is optimal due to the following example.

Example 4.2.9 ([IF00, List 16.6, No.88])**.** Consider the general weighted hypersurface $X_{42} \subset \mathbb{P}(1^2, 6, 14, 21)$, which is a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 2$. Then $\dim \overline{\varphi_{-6}(X_{42})} > 1$ while $\dim \overline{\varphi_{-5}(X_{42})} = 1$. So $\delta_1(X_{42}) = 6$.

Proof of Theorem 4.2.8. Since $P_{-1} > 0$, it is sufficient to prove that there exists an integer $m \leq 6$ such that $\dim \overline{\varphi_{-m}(X)} > 1$.

Assume, to the contrary, that $\delta_1(X) > 6$. Then, by applying Theorems 4.2.2 and 4.2.4 to the case $m = 1$, we have

$$P_{-1} = 2, P_{-2} = 3, P_{-3} = 4, P_{-4} = 5, P_{-5} = 6, P_{-6} = 7.$$

Now by those formulae in Subsection 4.2.2, we have $n^0_{1,2} = 1$, $n^0_{1,3} = 1$, $n^0_{1,4} = \epsilon_5 = 1 - \sigma_5$, and $0 = \epsilon_6 = 1 - \epsilon$. Hence $\epsilon = 1$, and this implies $\sigma_5 = n^0_{1,5} = 1$. Hence the basket $B^{(5)} = B^{(0)} = \{(1, 2), (1, 3), (1, 5)\}$ by $\epsilon_5 = 0$. Since $B^{(5)}$ admits no prime packings, $B = B^{(5)}$ and $-K^3_X = -K^3(\mathbb{B}(X)) = -1/30 < 0$, a contradiction. $\square$

4.2.4 $\mathbb{Q}$-Fano 3-folds with $h^0(-K) = 1$

We are going to prove the following theorem.

Theorem 4.2.10. *Let X be a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 1$. Then, for any integer $m \geq 9$, $\dim \overline{\varphi_{-m}(X)} > 1$. In particular, $\delta_1(X) \leq 9$.*

This result is optimal as well due to the following example.

Example 4.2.11 ([IF00, List 16.7, No.85])**.** Consider the general codimension 2 weighted complete intersection $X = X_{24,30} \subset \mathbb{P}(1, 8, 9, 10, 12, 15)$ which is a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 1$. Then $\dim \overline{\varphi_{-9}(X)} > 1$ and $\dim \overline{\varphi_{-8}(X)} = 1$ since $P_{-8} = 2$. So $\delta_1(X) = 9$.

Proof of Theorem 4.2.10. Since $P_{-1} > 0$, it is sufficient to prove that there exists an integer $m \leq 9$ such that $\dim \overline{\varphi_{-m}(X)} > 1$. Assume, to the contrary, that $\delta_1(X) > l$ for some integer $l \leq 9$. We will deduce a contradiction.

Applying Theorems 4.2.2 and 4.2.4 to the case $m = 1$, we distinguish the number n_0 (defined in Theorem 4.2.4). By Chen–Chen [CC08, Theorem 1.1], we have $n_0 \le 8$.

If $n_0 = 2$ and set $l = 6$, then Theorem 4.2.4(i)($m = 1$) implies that

$$P_{-1} = 1,\ P_{-2} = P_{-3} = 2,\ P_{-4} = P_{-5} = 3,\ P_{-6} = 4.$$

Then $n^0_{1,2} = 5$, $n^0_{1,3} = 1$, $n^0_{1,4} = \epsilon_5 = 1 - \sigma_5$, $0 = \epsilon_6 = 1 - \epsilon$. Hence $\epsilon = 1$, and this implies $\sigma_5 = n^0_{1,5} = 1$. Hence the basket $B^{(5)} = B^{(0)} = \{5 \times (1, 2), (1, 3), (1, 5)\}$ by $\epsilon_5 = 0$. Since $B^{(5)}$ admits no further prime packings, $B = B^{(5)}$ and $-K^3(\mathbb{B}) = -\frac{1}{30} < 0$, a contradiction. Thus $\delta_1(X) \le 6$.

If $n_0 = 3$ and set $l = 6$, then Theorem 4.2.4(i)($m = 1$) implies that

$$P_{-1} = P_{-2} = 1,\ P_{-3} = P_{-4} = P_{-5} = 2,\ P_{-6} = 3.$$

Then $n^0_{1,2} = 1$, $n^0_{1,3} = 4$, $n^0_{1,4} = \epsilon_5 = 1 - \sigma_5$, $0 = \epsilon_6 = 1 - \epsilon$. Hence $\epsilon = 1$, and this implies $\sigma_5 = n^0_{1,5} = 1$. Hence the basket $B^{(5)} = B^{(0)} = \{(1, 2), 4 \times (1, 3), (1, 5)\}$ by $\epsilon_5 = 0$. Since $B^{(5)}$ admits no further prime packings, $B = B^{(5)}$ and $-K^3(\mathbb{B}) = -\frac{1}{30} < 0$, a contradiction. Thus $\delta_1(X) \le 6$.

If $n_0 = 4$ and set $l = 6$, then Theorem 4.2.4(i)($m = 1$) implies that

$$P_{-1} = P_{-2} = P_{-3} = 1, P_{-4} = P_{-5} = P_{-6} = 2.$$

Then $n^0_{1,2} = 2$, $n^0_{1,3} = 1$, $n^0_{1,4} = 3 - \sigma_5$, $\epsilon_5 = 1 - \sigma_5$, $0 = \epsilon_6 = 1 - \epsilon$. Hence $\epsilon = 1$, and this implies $\sigma_5 = n^0_{1,5} = 1$. Hence $B^{(5)} = \{2 \times (1, 2), (1, 3), 2 \times (1, 4), (1, 5)\}$ by $\epsilon_5 = 0$. Hence $\epsilon_7 \le 1$ and $\epsilon_8 = 0$ by considering possible prime packings of $B^{(5)}$. On the other hand, $\epsilon_7 = P_{-7} - 1$ and $\epsilon_8 = P_{-8} - P_{-7}$. So $P_{-8} = \epsilon_7 + 1 \le 2$. But this contradicts $P_{-4} = 2$ and inequality (4.2.5). So $\delta_1(X) \le 6$.

If $n_0 = 5$ and set $l = 7$, then Theorem 4.2.4(i)($m = 1$) implies that

$$P_{-1} = P_{-2} = P_{-3} = P_{-4} = 1,\ P_{-5} = P_{-6} = P_{-7} = 2.$$

Then $n^0_{1,2} = 2$, $n^0_{1,3} = 2$, $n^0_{1,4} = 2 - \sigma_5$, $\epsilon_5 = 3 - \sigma_5$, $0 = \epsilon_6 = 2 - \epsilon$, $\epsilon_7 = 1 - 2\sigma_5 + 2n^0_{1,5} + n^0_{1,6}$. Hence $\epsilon = 2$, and this implies $(\sigma_5, n^0_{1,5}) = (1, 0)$ or $(2, 2)$. If $(\sigma_5, n^0_{1,5}) = (1, 0)$, then $n^0_{1,6} = 1$ by $\epsilon_7 \ge 0$. Hence $\epsilon_5 = 2$ and $B^{(5)} = \{2 \times (2, 5), (1, 4), (1, 6)\}$. Since $B^{(5)}$ admits no further prime packings, $B = B^{(5)}$ and $-K^3(\mathbb{B}) = -\frac{1}{60} < 0$, a contradiction. If $(\sigma_5, n^0_{1,5}) = (2, 2)$, then $\epsilon_5 = 1$, $\epsilon_7 = 1$, and $B^{(7)} = \{(3, 7), (1, 3), 2 \times (1, 5)\}$. Since $B^{(7)}$ admits no further prime packings, $B = B^{(7)}$ and $-K^3(\mathbb{B}) = -\frac{2}{105} < 0$, a contradiction. So $\delta_1(X) \le 7$.

If $n_0 = 6$ and set $l = 8$, then Theorem 4.2.4(i)($m = 1$) implies that

$$P_{-1} = P_{-2} = P_{-3} = P_{-4} = P_{-5} = 1, \ P_{-6} = P_{-7} = P_{-8} = 2.$$

Then $n^0_{1,2} = 2$, $n^0_{1,3} = 2$, $n^0_{1,4} = 2 - \sigma_5$, $\epsilon_5 = 2 - \sigma_5$, $0 = \epsilon_6 = 3 - \epsilon$. Hence $\epsilon = 3$ and $\sigma_5 \le 2$, and this implies $(\sigma_5, n^0_{1,5}) = (2,1)$. Then $\epsilon_5 = 0$ and $B^{(5)} = \{2 \times (1,2), 2 \times (1,3), (1,5), (1,s')\}$ for some $s' \ge 6$. This implies $\epsilon_7 = \epsilon_8 = 0$ since there are no further packings. On the other hand, $\epsilon_7 = 2 - 2\sigma_5 + 2n^0_{1,5} + n^0_{1,6}$ and $\epsilon_8 = 2 - 3\sigma_5 + 3n^0_{1,5} + 2n^0_{1,6} + n^0_{1,7}$. Hence $n^0_{1,6} = 0$, $n^0_{1,7} = 1$, and $B^{(7)} = \{2\times(1,2), 2\times(1,3), (1,5), (1,7)\}$. Since $B^{(7)}$ is minimal, $B = B^{(7)}$ and $-K^3(\mathbb{B}) = -\frac{1}{105} < 0$, a contradiction. Thus $\delta_1(X) \le 8$.

If $n_0 \ge 7$ and set $l = 9$, then Theorem 4.2.4(i)($m = 1$) implies that

$$P_{-1} = P_{-2} = P_{-3} = P_{-4} = P_{-5} = P_{-6} = 1, \ P_{-8} = P_{-9} = 2.$$

Then $n^0_{1,2} = 2$, $n^0_{1,3} = 2$, $n^0_{1,4} = 2-\sigma_5$, $\epsilon_5 = 2-\sigma_5$, $0 = \epsilon_6 = 2-\epsilon$. Hence $\epsilon = 2$ and $\sigma_5 \le 2$, and this implies $(\sigma_5, n^0_{1,5}) = (1,0)$ or $(2,2)$. If $(\sigma_5, n^0_{1,5}) = (2,2)$, then $B^{(5)} = \{2 \times (1,2), 2 \times (1,3), 2 \times (1,5)\}$ by $\epsilon_5 = 0$. Since $B^{(5)}$ admits no further prime packings, $B = B^{(5)}$ and $-K^3(\mathbb{B}) < 0$, a contradiction.

Thus we are left to consider the case: $(\sigma_5, n^0_{1,5}) = (1,0)$. Then we have $B^{(5)} = \{(1,2), (2,5), (1,3), (1,4), (1,s')\}$ with $s' \ge 6$ by $\epsilon_5 = 1$. Assume that $s' = 6$, 7. Clearly any basket B, with such a given $B^{(5)}$ dominates one of the following minimal ones:

$$B_1 = \{(3,7), (2,7), (1,s')\};$$
$$B_2 = \{(1,2), (3,8), (1,4), (1,s')\}.$$

Since $\sigma'(B) \ge \sigma'(B_i) \ge 2$ where $s' = 6,7$ and $i = 1,2$, inequality (4.2.4) fails for all B, which says that this case does not happen. Hence $s' \ge 8$, then the expression of ϵ_8 gives

$$P_{-8} - P_{-7} = \epsilon_8 + 1.$$

Hence $P_{-7} = P_{-6} = 1$ and $\epsilon_7 = \epsilon_8 = 0$ since $P_{-8} = 2$. We have $B^{(8)} = B^{(5)} = \{(1,2), (2,5), (1,3), (1,4), (1,s')\}$ with $s' \ge 8$. Since $B^{(8)}$ admits no further prime packings, $B = B^{(8)}$. By inequalities (4.2.3) and (4.2.4), s' can only be $9, 10, 11$. But then direct calculations show that $P_{-9} = 3$ in all these three cases, a contradiction. We have proved $\delta_1(X) \le 9$.

So we conclude the theorem. □

4.2.5 $\mathbb{Q}$-Fano 3-folds with $h^0(-K) = 0$

In this subsection we prove the following theorem.

Theorem 4.2.12. *Let X be a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 0$. Then there exists an integer $m_1 \leq 11$ such that $\dim \overline{\varphi_{-m_1}(X)} > 1$. Moreover, we can take such a number $m_1 \leq 8$ except for the following baskets of singularities:*

$$
\begin{array}{ll}
\text{No.1.} & \{2 \times (1,2), 3 \times (2,5), (1,3), (1,4)\}; \\
\text{No.2.} & \{5 \times (1,2), 2 \times (1,3), (2,7), (1,4)\}; \\
\text{No.3.} & \{5 \times (1,2), 2 \times (1,3), (3,11)\}; \\
\text{No.4.} & \{5 \times (1,2), (1,3), (3,10), (1,4)\}; \\
\text{No.A.} & \{7 \times (1,2), (3,7), (1,5)\}; \\
\text{No.B.} & \{6 \times (1,2), (4,9), (1,5)\}; \\
\text{No.C.} & \{5 \times (1,2), (5,11), (1,5)\}; \\
\text{No.D.} & \{4 \times (1,2), (6,13), (1,5)\}; \\
\text{No.E.} & \{7 \times (1,2), (3,8), (1,5)\}; \\
\text{No.F.} & \{5 \times (1,2), (4,9), (1,3), (1,5)\}.
\end{array}
$$

Remark 4.2.13. We do not know if this result is optimal since very few examples with $P_{-1} = 0$ are known. There are 4 known examples due to Iano-Fletcher [IF00, List 16.7, No.60] and Altinok–Reid [AR], [Reid00, Example 9.14]. For these examples we can see that $\dim \overline{\varphi_{-8}(X)} > 1$ by our theorem. Moreover, in next subsection we will treat the exceptional cases. If one can confirm either the existence or non-existence of type No.1–No.4, the result becomes optimal and so does Theorem 1.2.9.

Before proving Theorem 4.2.12, we recall a result by J. A. Chen and M. Chen.

Proposition 4.2.14 ([CC08, Theorem 3.5]). *Any geometric basket of weak $\mathbb{Q}$-Fano 3-folds with $P_{-1} = P_{-2} = 0$ is among the following list:*

	B	$-K^3$	P_{-3}	P_{-4}	P_{-5}	P_{-6}	P_{-7}	P_{-8}
No.1.	$\{2 \times (1,2), 3 \times (2,5), (1,3), (1,4)\}$	1/60	0	0	1	1	1	2
No.2.	$\{5 \times (1,2), 2 \times (1,3), (2,7), (1,4)\}$	1/84	0	1	0	1	1	2
No.3.	$\{5 \times (1,2), 2 \times (1,3), (3,11)\}$	1/66	0	1	0	1	1	2
No.4.	$\{5 \times (1,2), (1,3), (3,10), (1,4)\}$	1/60	0	1	0	1	1	2
No.5.	$\{5 \times (1,2), (1,3), 2 \times (2,7)\}$	1/42	0	1	0	1	2	3
No.6.	$\{4 \times (1,2), (2,5), 2 \times (1,3), 2 \times (1,4)\}$	1/30	0	1	1	2	2	4
No.7.	$\{3 \times (1,2), (2,5), 5 \times (1,3)\}$	1/30	1	1	1	3	3	4
No.8.	$\{2 \times (1,2), (3,7), 5 \times (1,3)\}$	1/21	1	1	1	3	4	5
No.9.	$\{(1,2), (4,9), 5 \times (1,3)\}$	1/18	1	1	1	3	4	5
No.10.	$\{3 \times (1,2), (3,8), 4 \times (1,3)\}$	1/24	1	1	1	3	3	5
No.11.	$\{3 \times (1,2), (4,11), 3 \times (1,3)\}$	1/22	1	1	1	3	3	5
No.12.	$\{3 \times (1,2), (5,14), 2 \times (1,3)\}$	1/21	1	1	1	3	3	5
No.13.	$\{2 \times (1,2), 2 \times (2,5), 4 \times (1,3)\}$	1/15	1	1	2	4	5	7
No.14.	$\{(1,2), (3,7), (2,5), 4 \times (1,3)\}$	17/210	1	1	2	4	6	8
No.15.	$\{2 \times (1,2), (2,5), (3,8), 3 \times (1,3)\}$	3/40	1	1	2	4	5	8
No.16.	$\{2 \times (1,2), (5,13), 3 \times (1,3)\}$	1/13	1	1	2	4	5	8
No.17.	$\{(1,2), 3 \times (2,5), 3 \times (1,3)\}$	1/10	1	1	3	5	7	10
No.18.	$\{4 \times (1,2), 5 \times (1,3), (1,4)\}$	1/12	1	2	2	5	6	9
No.19.	$\{4 \times (1,2), 4 \times (1,3), (2,7)\}$	2/21	1	2	2	5	7	10
No.20.	$\{4 \times (1,2), 3 \times (1,3), (3,10)\}$	1/10	1	2	2	5	7	10
No.21.	$\{3 \times (1,2), (2,5), 4 \times (1,3), (1,4)\}$	7/60	1	2	3	6	8	12
No.22.	$\{3 \times (1,2), 7 \times (1,3)\}$	1/6	2	3	4	9	12	17
No.23.	$\{2 \times (1,2), (2,5), 6 \times (1,3)\}$	1/5	2	3	5	10	14	20

Proof of Theorem 4.2.12. In the proof, we will always take a suitable integer m satisfying one of the conditions in Theorem 4.2.2. If necessary, we apply Theorem 4.2.4 on m and take $m_1 = l_0 m$.

Case I. $P_{-2} = 0$.

The basket $B = B_X$ of the singularities of X is among the list of Proposition 4.2.14. We just discuss it case by case.

If B is of type No.1, take $m = 5$. Since $P_{-5} = 1$ and $P_{-10} = 4$, we can take $m_1 = 10$.

If B is of type No.2, take $m = 11$. Since $P_{-11} = 4$, we can take $m_1 = 11$.

If B is of type No.3, take $m = 10$. Since $P_{-10} = 3$, we can take $m_1 = 10$.

If B is of type No.4, take $m = 11$. Since $P_{-11} = 4$, we can take $m_1 = 11$.

If B is of type No.5, take $m = 8$. Since $P_{-8} = 3$, we can take $m_1 = 8$.

If B is of type No.6, take $m = 8$. Since $P_{-8} = 4$, we can take $m_1 = 8$.

If B is of type No.7-No.21, take $m = 3$. Since $P_{-3} = 1$ and $P_{-6} \geq 3$, we can take $m_1 = 6$.

If B is of type No.22-No.23, take $m = 3$. Since $P_{-3} = 2$ and $P_{-6} \geq 9$, we can take $m_1 = 6$.

Case II. $P_{-2} > 0$.

Since $P_{-1} = 0$, the basket $B^{(0)}$ has datum

$$\begin{cases} n^0_{1,2} = 5 + 4P_{-2} - P_{-3}; \\ n^0_{1,3} = 4 - 2P_{-2} + 3P_{-3} - P_{-4}; \\ n^0_{1,4} = 1 - P_{-2} - 2P_{-3} + P_{-4} - \sigma_5. \end{cases}$$

By Lemma 4.2.7, $B^{(0)}$ satisfies inequality (4.2.3) and thus

$$\begin{aligned} 0 < \gamma(B^{(0)}) &= \sum_{r\geq 2}(\frac{1}{r} - r)n^0_{1,r} + 24 \\ &\leq \sum_{r=2,3,4}(\frac{1}{r} - r)n^0_{1,r} - \frac{24}{5}\sigma_5 + 24 \\ &= \frac{25}{12} + \frac{37}{12}P_{-2} + P_{-3} - \frac{13}{12}P_{-4} - \frac{21}{20}\sigma_5. \end{aligned}$$

Hence, by $n^0_{1,3} \geq 0$ and $n^0_{1,4} \geq 0$, we have

$$\begin{cases} \frac{25}{12} + \frac{37}{12}P_{-2} + P_{-3} - \frac{13}{12}P_{-4} - \frac{21}{20}\sigma_5 > 0; & (4.2.6) \\ 4 - 2P_{-2} + 3P_{-3} - P_{-4} \geq 0; & (4.2.7) \\ 1 - P_{-2} - 2P_{-3} + P_{-4} - \sigma_5 \geq 0. & (4.2.8) \end{cases}$$

Considering the inequality "(4.2.6)+(4.2.7)+2×(4.2.8)":

$$\frac{97}{12} - \frac{11}{12}P_{-2} - \frac{1}{12}P_{-4} - \frac{61}{20}\sigma_5 > 0, \qquad (4.2.9)$$

we obtain $\sigma_5 \le 2$.

Subcase II-1. $\sigma_5 = 0$.

At first, we consider the case $P_{-3} = 0$. By inequality (4.2.7), we have $2P_{-2} + P_{-4} \le 4$. Since $1 \le P_{-2} \le P_{-4}$, it follows that $(P_{-2}, P_{-4}) = (1,1)$ or $(1,2)$. If $(P_{-2}, P_{-4}) = (1,1)$, then $B^{(0)} = \{9 \times (1,2), (1,3), (1,4)\}$ with $-K^3(B^{(0)}) = -\frac{1}{12} < 0$. Considering a minimal basket $B_{\min}$ dominated by $B^{(0)}$, then either $B_{\min} = \{(10,21), (1,4)\}$ with $-K^3(B_{\min}) = -\frac{1}{84} < 0$ or $B_{\min} = \{9 \times (1,2), (2,7)\}$ with $-K^3(B_{\min}) = -\frac{1}{14} < 0$. Thus $-K^3(B) \le -K^3(B_{\min}) < 0$, a contradiction. If $(P_{-2}, P_{-4}) = (1,2)$, then $B^{(0)} = \{9 \times (1,2), 2 \times (1,4)\}$. Since $B^{(0)}$ admits no prime packings anymore, $B = B^{(0)}$ and $-K^3(B) = 0$, a contradiction.

Let us consider the case $P_{-3} \ge 1$. Since $\sigma_5 = 0$, $B^{(0)}$ is composed of $(1,2), (1,3), (1,4)$. In particular, $4b \ge r$ holds for every pair $(b,r) \in B^{(0)}$. As an easy conclusion, after packings, $4b \ge r$ holds for every pair $(b,r) \in B$. So $m = 3$ satisfies the condition of Theorem 4.2.2. By Theorem 4.2.4, we can take $m_1 = 3$ or 6 unless $(P_{-3}, P_{-6}) = (1,1), (1,2), (2,3)$. By inequality (4.2.8),

$$P_{-4} \ge 2P_{-3} + P_{-2} - 1 \ge 2P_{-3}. \tag{4.2.10}$$

By $P_{-2} > 0$, $P_{-6} \ge P_{-4}$. Thus we only need to consider the case $(P_{-3}, P_{-6}) = (1,2)$. By inequality (4.2.10), $P_{-2} = 1$ and $P_{-4} = 2$. On the other hand,

$$0 = \epsilon_6 = 3P_{-1} + P_{-2} - P_{-3} - P_{-4} - P_{-5} + P_{-6} - \epsilon = -P_{-5}.$$

This implies $P_{-5} = 0$ which contradicts $P_{-2} = P_{-3} = 1$.

Subcase II-2. $\sigma_5 = 2$.

By inequality (4.2.9) and $P_{-4} \ge 2P_{-2} - 1$, we have $P_{-2} \le 1$. Hence $P_{-2} = 1$ and, by inequalities (4.2.6)-(4.2.8), we have inequalities:

$$\begin{cases} \frac{46}{15} + P_{-3} - \frac{13}{12}P_{-4} > 0; & (4.2.11)\\ 2 + 3P_{-3} - P_{-4} \ge 0; & (4.2.12)\\ -2 - 2P_{-3} + P_{-4} \ge 0. & (4.2.13) \end{cases}$$

Considering the inequality "$2 \times (4.2.11) + (4.2.13)$", we have $P_{-4} \le 3$. Hence $P_{-3} = 0$ by inequality (4.2.13), and $P_{-4} = 2$ by inequalities (4.2.12) and (4.2.13). Then $B^{(0)} = \{9 \times (1,2), (1,s_1), (1,s_2)\}$ with $5 \le s_1 \le s_2$. If $s_2 > 5$, then $\gamma(B^{(0)}) \le 9 \times (\frac{1}{2} - 2) + (\frac{1}{5} - 5) + (\frac{1}{6} - 6) + 24 < 0$, a contradiction. Thus $B^{(0)} = \{9 \times (1,2), 2 \times (1,5)\}$. Since $B^{(0)}$ admits no further prime packings, $B = B^{(0)}$. Take $m = 5$. Since $P_{-5} = 3$ by $\epsilon_5 = 0$, we can take $m_1 = 5$ by Theorem 4.2.4.

Subcase II-3. $\sigma_5 = 1$.

By inequalities (4.2.6)-(4.2.8), we have

$$\begin{cases} 12 + 37P_{-2} + 12P_{-3} - 13P_{-4} \geq 0; & (4.2.14) \\ 4 - 2P_{-2} + 3P_{-3} - P_{-4} \geq 0; & (4.2.15) \\ -P_{-2} - 2P_{-3} + P_{-4} \geq 0. & (4.2.16) \end{cases}$$

Considering the inequality "(4.2.14) + 13 × (4.2.16)", we have

$$7P_{-3} \leq 12P_{-2} + 6. \tag{4.2.17}$$

Considering the inequality "(4.2.15) + (4.2.16)", we have

$$3P_{-2} \leq P_{-3} + 4. \tag{4.2.18}$$

Inequalities (4.2.17) and (4.2.18) imply $P_{-2} \leq 3$.

Subsubcase II-3-i. $(\sigma_5, P_{-2}) = (1, 3)$.

By inequalities (4.2.17) and (4.2.18), $5 \leq P_{-3} \leq 6$.

If $P_{-3} = 6$, by inequalities (4.2.14) and (4.2.16), $P_{-4} = 15$. Then $B^{(0)} = \{11 \times (1,2), (1,3), (1,s)\}$ for some integer $s \geq 5$. By $\gamma(B^{(0)}) > 0$, we have $s = 5$. Since the one-step packing $B_1 = \{10 \times (1,2), (2,5), (1,5)\}$ has negative $\gamma(B_1)$, $B = B^{(0)} = \{11 \times (1,2), (1,3), (1,5)\}$. Take $m = 4$. Since $P_{-4} = 15$, we can take $m_1 = 4$ by Theorem 4.2.4.

If $P_{-3} = 5$, by inequalities (4.2.15) and (4.2.16), $P_{-4} = 13$. Then $B^{(0)} = \{12 \times (1,2), (1,s)\}$ for some integer $s \geq 5$. By $\gamma(B^{(0)}) > 0$, we have $s = 5, 6$. Clearly $B = B^{(0)}$. Take $m = 5$. Since $P_{-5} = 22$, we can take $m_1 = 5$ by Theorem 4.2.4.

Subsubcase II-3-ii. $(\sigma_5, P_{-2}) = (1, 2)$.

By inequalities (4.2.17) and (4.2.18), $2 \leq P_{-3} \leq 4$.

If $P_{-3} = 4$, by inequalities (4.2.14) and (4.2.16), $P_{-4} = 10$. Then $B^{(0)} = \{9 \times (1,2), 2 \times (1,3), (1,s)\}$ for some integer $s \geq 5$. By $\gamma(B^{(0)}) > 0$, we have $s = 5$. Since the one-step packing $B_1 = \{8 \times (1,2), (2,5), (1,3), (1,5)\}$ has negative $\gamma(B_1)$, $B = B^{(0)} = \{9 \times (1,2), 2 \times (1,3), (1,5)\}$. Take $m = 4$. Since $P_{-4} = 10$, we can take $m_1 = 4$ by Theorem 4.2.4.

If $P_{-3} = 3$, by inequalities (4.2.15) and (4.2.16), $8 \leq P_{-4} \leq 9$. Firstly let us consider the case $P_{-4} = 9$. Clearly $B^{(0)} = \{10 \times (1,2), (1,4), (1,s)\}$ for some integer $s \geq 5$. By $\gamma(B^{(0)}) > 0$, we have $s = 5$. If $B = B^{(0)}$, we may take $m = 4$. Since $P_{-4} \geq 9$, we can take $m_1 = 4$ by Theorem 4.2.4. If $B \neq B^{(0)}$, we have $B = \{10 \times (1,2), (2,9)\}$. Take $m = 8$. Since $P_{-8} \geq 3$, we can take $m_1 = 8$ by Theorem 4.2.4. Now we consider the case $P_{-4} = 8$. We have $B^{(0)} = \{10 \times (1,2), (1,3), (1,s)\}$ for some integer $s \geq 5$. Since $\gamma(B^{(0)}) > 0$, we have $5 \leq s \leq 6$. For the case $(P_{-4}, s) = (8, 6)$, we get

$B = \{10\times(1,2),(1,3),(1,6)\}$ since any possible packing of $B^{(0)}$ has negative γ. Take $m = 5$. Since $P_{-5} = 13$, we can take $m_1 = 5$ by Theorem 4.2.4. For the case $(P_{-4}, s) = (8, 5)$, we get either $B = \{10 \times (1,2),(1,3),(1,5)\}$ or $B = \{9 \times (1,2),(2,5),(1,5)\}$ or $B = \{8 \times (1,2),(3,7),(1,5)\}$ by $\gamma > 0$. For all these cases, take $m = 6$. Since $P_{-6} \geq 3$, we can take $m_1 = 6$ by Theorem 4.2.4.

If $P_{-3} = 2$, we have $P_{-4} = 6$ by inequalities (4.2.15) and (4.2.16). Then $B^{(0)} = \{11 \times (1,2),(1,s)\}$ for some integer $s \geq 5$. Similarly, $\gamma(B^{(0)}) > 0$ implies $5 \leq s \leq 7$. Since $B^{(0)}$ admits no further prime packings, $B = B^{(0)}$. Take $m = 6$. Since $P_{-6} \geq 3$, we can take $m_1 = 6$ by Theorem 4.2.4.

Subsubcase II-3-iii. $(\sigma_5, P_{-2}) = (1, 1)$.

By inequality (4.2.17), $P_{-3} \leq 2$.

If $P_{-3} = 2$, we have $P_{-4} = 5$ by inequalities (4.2.14) and (4.2.16). Then $B^{(0)} = \{7\times(1,2),3\times(1,3),(1,s)\}$ for some integer $s \geq 5$. Similarly, $\gamma(B^{(0)}) > 0$ implies $s = 5$. Furthermore, we have either $B = \{7\times(1,2),3\times(1,3),(1,5)\}$ or $B = \{6 \times (1,2),(2,5),2 \times (1,3),(1,5)\}$ by $\gamma > 0$. Take $m = 4$. Since $P_{-4} = 5$, we can take $m_1 = 4$ by Theorem 4.2.4.

If $P_{-3} = 1$, we have $3 \leq P_{-4} \leq 4$ by inequalities (4.2.14) and (4.2.16). Consider the case $(P_{-3}, P_{-4}) = (1, 4)$. We have $B^{(0)} = \{8\times(1,2),(1,3),(1,4),(1,s)\}$ for some integer $s \geq 5$. Again we have $s = 5$ since $\gamma(B^{(0)}) > 0$. With the property $\gamma > 0$ and considering all possible baskets with $B^{(0)}$, we see that B must be one of the following baskets:

$$
\begin{aligned}
B_1 &= \{8 \times (1,2),(1,3),(1,4),(1,5)\},\\
B_2 &= \{8 \times (1,2),(2,7),(1,5)\},\\
B_3 &= \{8 \times (1,2),(1,3),(2,9)\},\\
B_4 &= \{7 \times (1,2),(2,5),(1,4),(1,5)\}.
\end{aligned}
$$

For B_2, take $m = 6$. Since $P_{-6}(B_2) \geq 3$, we can take $m_1 = 6$ by Theorem 4.2.4. For B_3, take $m = 8$. Since $P_{-8}(B_3) \geq 3$, we can take $m_1 = 8$ by Theorem 4.2.4. For B_1 and B_4, take $m = 4$. Similarly we can take $m_1 = 4$ by Theorem 4.2.4. Consider the case $(P_{-3}, P_{-4}) = (1, 3)$. We have $B^{(0)} = \{8 \times (1,2), 2 \times (1,3),(1,s)\}$ for some integer $s \geq 5$. Similarly, $\gamma(B^{(0)}) > 0$ implies $5 \leq s \leq 6$. If $s = 6$, we see either $B = \{8 \times (1,2), 2 \times (1,3),(1,6)\}$ or $B = \{7 \times (1,2),(2,5),(1,3),(1,6)\}$ since $\gamma(B) > 0$. Take $m = 5$. Since $P_{-5} \geq 3$, we can take $m_1 = 5$ by Theorem 4.2.4. If $s = 5$, by considering all possible packings dominated by $B^{(0)}$ and using the property $\gamma > 0$, we see that B must be one of the following baskets:

$$
\begin{aligned}
B_i &= \{8 \times (1,2), 2 \times (1,3),(1,5)\},\\
B_{ii} &= \{7 \times (1,2),(2,5),(1,3),(1,5)\},
\end{aligned}
$$

$$
\begin{aligned}
B_{iii} &= \{6 \times (1,2), 2 \times (2,5), (1,5)\}, \\
B_{iv} &= \{6 \times (1,2), (3,7), (1,3), (1,5)\}, \\
B_{v} &= \{5 \times (1,2), (4,9), (1,3), (1,5)\}, \\
B_{vi} &= \{7 \times (1,2), (3,8), (1,5)\}, \\
B_{vii} &= \{5 \times (1,2), (3,7), (2,5), (1,5)\}.
\end{aligned}
$$

For B_v (corresponding to No.F) and B_{vi} (corresponding to No.E), take $m = 9$. Since $P_{-9} \geq 3$, we can take $m_1 = 9$ by Theorem 4.2.4. For other cases, take $m = 6$. Since $P_{-6} \geq 3$, we can take $m_1 = 6$ by Theorem 4.2.4.

If $P_{-3} = 0$, by inequality (4.2.15), $P_{-4} \leq 2$. Consider the case $(P_{-3}, P_{-4}) = (0, 2)$. We have $B^{(0)} = \{9 \times (1,2), (1,4), (1,s)\}$ for some integer $s \geq 5$. In fact, $5 \leq s \leq 6$ by $\gamma(B^{(0)}) > 0$. When $s = 6$, $B = B^{(0)}$ since $B^{(0)}$ admits no further packings. Take $m = 7$. Since $P_{-7} = 6$, we can take $m_1 = 7$ by Theorem 4.2.4. When $s = 5$, the property $\gamma > 0$ implies that $B^{(0)}$ admits at most one further packings. Thus either $B = \{9 \times (1,2), (1,4), (1,5)\}$ (take $m = 4$) or $B = \{9 \times (1,2), (2,9)\}$ (take $m = 8$). For the first basket, $P_{-4} = 2$ and $P_{-8} = 7$, we can take $m_1 = 8$ by Theorem 4.2.4. For the second basket, $P_{-8} = 7$ and we can take $m_1 = 8$ by Theorem 4.2.4.

Finally we consider the case $(P_{-3}, P_{-4}) = (0, 1)$. We have $B^{(0)} = \{9 \times (1,2), (1,3), (1,s)\}$ for some integer $s \geq 5$. Similarly, $\gamma(B^{(0)}) > 0$ implies $5 \leq s \leq 7$. When $s = 7$, the property $\gamma > 0$ implies that either $B = \{9 \times (1,2), (1,3), (1,7)\}$ or $B = \{8 \times (1,2), (2,5), (1,7)\}$. Take $m = 8$. Since $P_{-8} \geq 3$, we can take $m_1 = 8$ by Theorem 4.2.4. When $s = 6$, the inequalities $\gamma > 0$ and $-K^3 > 0$ imply that B must be one of the following baskets:

$$
\begin{aligned}
&\{8 \times (1,2), (2,5), (1,6)\}, \\
&\{7 \times (1,2), (3,7), (1,6)\}, \\
&\{6 \times (1,2), (4,9), (1,6)\}.
\end{aligned}
$$

Take $m = 7$. Since $P_{-7} \geq 3$, we can take $m_1 = 7$ by Theorem 4.2.4. When $s = 5$, inequalities $\gamma > 0$ and $-K^3 > 0$ imply that B is among one of the following baskets:

$$
\begin{aligned}
B_a &= \{7 \times (1,2), (3,7), (1,5)\}, \\
B_b &= \{6 \times (1,2), (4,9), (1,5)\}, \\
B_c &= \{5 \times (1,2), (5,11), (1,5)\}, \\
B_d &= \{4 \times (1,2), (6,13), (1,5)\}.
\end{aligned}
$$

For B_d (corresponding to No.D), take $m = 11$. Since $P_{-11} \geq 3$, we can take $m_1 = 11$ by Theorem 4.2.4. For other baskets (corresponding to No.A, No.B, No.C), take $m = 9$. Since $P_{-9} \geq 3$, we can take $m_1 = 9$ by Theorem 4.2.4. So the theorem is proved. □

4.2.6 Exceptional cases

In this subsection, we treat the exceptional cases in Theorem 4.2.12.

Theorem 4.2.15. *Let X be a $\mathbb{Q}$-Fano 3-fold with basket of singularities B.*

(i) If B is of type No.1-No.4 as in Theorem 4.2.12, then $\dim \overline{\varphi_{-10}(X)} > 1$.

(ii) If B is of type No.A-No.D as in Theorem 4.2.12, then $\dim \overline{\varphi_{-8}(X)} > 1$.

(iii) If B is of type No.E-No.F as in Theorem 4.2.12, then $\dim \overline{\varphi_{-4}(X)} > 1$.

Proof. (i). Recall the proof in Case I of Theorem 4.2.12. We may only consider the two cases with No.2 and No.4. Since $P_{-9} = 2$, $\delta_1(X) \geq 10$. We want to show that $\delta_1(X) = 10$ in both cases. In fact, we have $P_{-4} = P_{-6} = 1$, $P_{-8} = 2$, $P_{-10} \geq 3$. Note that the conditions of Theorem 4.2.2 are all satisfied with $m = 4$. It follows that $-4K_X \sim E$ is a prime divisor. Assume that $\dim \overline{\varphi_{-10}(X)} = 1$, then we can write $|-10K_X| = |nS| + E'$ with $n \geq 2$, $|S|$ is an irreducible rational pencil of surfaces, and E' is the fixed part. By $P_{-6} > 0$, we have $E \leq |nS| + E'$. Since E is reduced and irreducible, either $E \leq |S|$ or $E \leq E'$ holds. Then

$$P_{-6} = h^0(-10K_X - E) = h^0(nS + E' - E) \geq h^0(S) = 2,$$

a contradiction.

(ii). Recall the last part of Subsubcase II-3-iii in the proof of Theorem 4.2.12. If B is of type No.A–No.D, we have $P_{-2} = P_{-4} = 1$, $P_{-6} = 2$, and $P_{-8} = 3$. Assume, to the contrary, that $\dim \overline{\varphi_{-8}(X)} = 1$.

Write $-2K_X \sim D$ for some effective Weil divisor. By Theorem 4.2.4(i) (with $m = 2$), D must be either reducible or non-reduced. As in the proof of Theorem 4.2.2, take E to be any strictly effective divisor such that $E < D$. Then inequality (4.2.1) must fail for some singularity Q in B_a–B_d. Clearly, such an offending singularity Q must be "$(1, 5)$". By equality (4.2.2), the local index $i_Q(E)$ of E should be 4 since inequality (4.2.1) holds for $i \in \{0, 1, 2, 3\}$ and $(b, r) = (1, 5)$, that is, $E \sim -K_X$ at Q. Since E is arbitrary such that $0 < E < D$ and $i_Q(-2K_X) = 3$, we conclude that $D = E_1 + E_2$ where E_i is fixed prime divisor with $i_Q(E_i) = 4$ for $i = 1, 2$.

If $E_1 = E_2$, then $2(-K_X - E_1) \sim 0$. By [Pro10, Proposition 2.9] and $-K_X - E_1$ is Cartier at Q, we conclude that $-K_X - E_1$ is not 2-torsion. Hence $-K_X - E_1 \sim 0$, which contradicts $P_{-1} = 0$. Thus E_1 and E_2 are different prime divisors.

Since $|-6K_X| \preceq |-8K_X|$, by Lemma 2.4.2 we can write

$$|-6K_X| = |S| + a_6E_1 + b_6E_2,$$

$$|-8K_X| = |2S| + a_8E_1 + b_8E_2,$$

where $|S|$ is an irreducible rational pencil of surfaces, $a_iE_1 + b_iE_2$ is the fixed part, $a_i, b_i \in \mathbb{N}$ for $i = 6, 8$.

Claim 6. $a_6b_6 = a_8b_8 = 0$.

Proof. Assume that $a_6, b_6 \geq 1$, then

$$P_{-4} = h^0(-6K_X - E_1 - E_2) \geq h^0(S) = 2,$$

a contradiction. Similarly, we have $a_8b_8 = 0$. □

We may assume that $b_6 = 0$. Then

$$3E_1 + 3E_2 \in |S + a_6E_1| = |S| + a_6E_1. \tag{4.2.19}$$

It follows that $a_6 \leq 3$.

Case ii.1. $b_8 = 0$.

In this case

$$2S + a_8E_1 \sim -8K_X \sim -6K_X + E_1 + E_2 \sim S + (a_6 + 1)E_1 + E_2.$$

Since a_8E_1 is the fixed part of $|2S + a_8E_1|$, $a_8 \leq a_6 + 1$. Then

$$S \sim (a_6 + 1 - a_8)E_1 + E_2. \tag{4.2.20}$$

Considering relations (4.2.19) and (4.2.20),

$$(2a_6 + 1 - a_8)E_1 + E_2 \sim 3E_1 + 3E_2. \tag{4.2.21}$$

Clearly, $2a_6+1-a_8 \leq 3$ is absurd. Thus $2a_6+1-a_8 \geq 4$. And $2a_6+1-a_8 \leq 7$ since $a_6 \leq 3$. Locally at Q, since $i_Q(E_1) = i_Q(E_2) = 4$, we have

$$2a_6 + 1 - a_8 \equiv 0 \mod 5.$$

So $2a_6 + 1 - a_8 = 5$. Then relation (4.2.21) implies $2E_1 \sim 2E_2$. By [Pro10, Proposition 2.9], we conclude that $E_1 \sim E_2$, a contradiction.

Case ii.2. $a_8 = 0$ and $b_8 > 0$.

In the case

$$2S + b_8E_2 \sim -8K_X \sim -6K_X + E_1 + E_2 \sim S + (a_6 + 1)E_1 + E_2.$$

This implies that $b_8 \leq 1$. Hence $b_8 = 1$ and

$$S \sim (a_6 + 1)E_1. \tag{4.2.22}$$

Considering relations (4.2.19) and (4.2.22),

$$(2a_6 + 1)E_1 \sim 3E_1 + 3E_2. \tag{4.2.23}$$

Clearly $2a_6 + 1 \geq 4$ and $2a_6 + 1 \leq 7$ since $a_6 \leq 3$. Locally at Q, since $i_Q(E_1) = i_Q(E_2) = 4$, we have

$$2a_6 + 1 \equiv 1 \mod 5.$$

Since $4 \leq 2a_6 + 1 \leq 7$, this is impossible.

(iii). Recall the cases with B_v (No.F) and B_{vi} (No.E) (see Subsubcase II-3-iii in the proof of Theorem 4.2.12). We have $P_{-2} = 1$, $P_{-4} = 3$. Assume, to the contrary, that $\dim \overline{\varphi_{-4}(X)} = 1$.

We can write $-2K_X \sim D$ for some effective divisor D. By the same argument as (ii), $D = E_1+E_2$ with E_i reduced and irreducible and $i_Q(E_i) = 4$ for $i = 1, 2$ where Q is the singularity "(1, 5)". Note that, however, we do not know if E_1 and E_2 are different.

We can write

$$|-4K_X| = |2S| + a_4E_1 + b_4E_2,$$

where $|S|$ is an irreducible rational pencil of surfaces and $a_4E_1 + b_4E_2$ is the fixed part, $a_4, b_4 \in \mathbb{N}$. Hence

$$2S + a_4E_1 + b_4E_2 \sim -4K_X \sim 2(-2K_X) \sim 2E_1 + 2E_2.$$

Since $a_4E_1 + b_4E_2$ is the fixed part of $|2S + a_4E_1 + b_4E_2|$, we may assume $a_4 \leq b_4 \leq 2$.

If $b_4 = 2$, then $2S \sim (2 - a_4)E_1$. Hence $E_1 \leq S$ by the irreducibility of E_1. Then

$$1 = h^0(E_1) \geq h^0(2S - E_1) \geq h^0(S) = 2,$$

a contradiction.

If $b_4 = 1 \geq a_4$, then $2S \sim (2 - a_4)E_1 + E_2 \geq E_1 + E_2$. Hence $E_1 \leq S$ by the irreducibility of E_1. Then

$$1 = h^0(E_1 + E_2) \geq h^0(2S - E_1) \geq h^0(S) = 2,$$

a contradiction.

Hence $a_4 = b_4 = 0$ and $-4K_X \sim 2S \sim 2E_1+2E_2$. Note that $h^0(E_1+E_2) = 1$ and

$$2E_1 + 2E_2 \in |2S| = |S| + |S|,$$

we have

$$S \sim 2E_1 \sim 2E_2.$$

Hence
$$4(-K_X - E_1) \sim 2(E_1 - E_2) \sim 0.$$
By [Pro10, Proposition 2.9], there are no 4-torsion Weil divisors. So
$$E_1 - E_2 \sim 2(-K_X - E_1) \sim 0.$$
Then
$$-2K_X \sim E_1 + E_2 \sim 2E_1 \sim S.$$
This contradicts $h^0(-2K_X) = 1$.

So we have proved the theorem. $\square$

To make the summary, Theorems 4.2.12 and 4.2.15 directly imply the following:

Corollary 4.2.16. *Let X be a $\mathbb{Q}$-Fano 3-fold with $P_{-1} = 0$. Then $\delta_1(X) \leq 8$ except for the following cases:*

No.1.	$\{2 \times (1,2), 3 \times (2,5), (1,3), (1,4)\}$	$\delta_1(X) = 10$;
No.2.	$\{5 \times (1,2), 2 \times (1,3), (2,7), (1,4)\}$	$\delta_1(X) = 10$;
No.3.	$\{5 \times (1,2), 2 \times (1,3), (3,11)\}$	$\delta_1(X) = 10$;
No.4.	$\{5 \times (1,2), (1,3), (3,10), (1,4)\}$	$\delta_1(X) = 10$;
No.A.	$\{7 \times (1,2), (3,7), (1,5)\}$	$\delta_1(X) = 8$;
No.B.	$\{6 \times (1,2), (4,9), (1,5)\}$	$\delta_1(X) = 8$;
No.C.	$\{5 \times (1,2), (5,11), (1,5)\}$	$\delta_1(X) = 8$;
No.D.	$\{4 \times (1,2), (6,13), (1,5)\}$	$\delta_1(X) = 8$;
No.E.	$\{7 \times (1,2), (3,8), (1,5)\}$	$\delta_1(X) = 4$;
No.F.	$\{5 \times (1,2), (4,9), (1,3), (1,5)\}$	$\delta_1(X) = 4$.

Theorem 1.2.9 follows directly from Theorems 4.2.8 and 4.2.10, and Corollaries 4.2.3 and 4.2.16.

4.3 When is $|-mK_X|$ not composed with a pencil (Part II)?

As we have seen in the last section, the condition $\rho(X) = 1$ is crucial to proving Theorem 4.2.2. For arbitrary weak $\mathbb{Q}$-Fano 3-folds, we have to study in an alternative way. Naturally what we can prove is weaker than Theorem 1.2.9.

Let X be a weak $\mathbb{Q}$-Fano 3-fold. We are going to estimate $\delta_1(X)$ from above. The main idea is to relate this problem to the value distribution of the Hilbert polynomial $\chi_{-m} = P_{-m}$. Recall that this is done by Proposition 2.6.2, which gives the following corollary.

Corollary 4.3.1. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. If*

$$P_{-m} > r_X(-K_X)^3 m + 1$$

for some integer m, then $|-mK_X|$ is not composed with a pencil.

Next we estimate the number m which satisfies Corollary 4.3.1. We will do this in two steps as follows.

Proposition 4.3.2. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Take an arbitrary real number $0 < t \leq 37$. Denote $r_{\max} := \max\{r_i \in B_X\}$ the maximum of local indices of singularities. If*

$$m \geq \max\left\{37, \frac{r_{\max} t}{3}, \sqrt{6r_X + \frac{12}{t(-K_X^3)}}\right\},$$

then $P_{-m} \geq r_X(-K_X^3)m + 2$. In particular, $|-mK_X|$ is not composed with a pencil.

Proof. By Reid's formula, there exists a basket of singularities

$$B_X = \{(b_i, r_i) \mid i = 1, \dots, s; 0 < b_i \leq \frac{r_i}{2}; b_i \text{ is coprime to } r_i\}$$

such that we have the formula

$$P_{-n} = \frac{1}{12} n(n+1)(2n+1)(-K_X^3) + 2n + 1 - l(-n)$$

for any $n > 0$, where

$$l(-n) = \sum_i \sum_{j=1}^{n} \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i}.$$

To estimate the lower bound of P_{-n}, we need to bound $l(-n)$ from above.

For any pair $(b, r) \in B_X$, we have $r \leq 24$ by inequality (4.1.1). In fact, we have the following estimation.

1. If $r = 2$, then
$$\frac{\overline{jb}(r - \overline{jb})}{2r} = \begin{cases} \frac{1}{4}, & \text{if } j \text{ odd;} \\ 0, & \text{if } j \text{ even.} \end{cases}$$

2. If r is odd, then $\frac{\overline{jb}(r-\overline{jb})}{2r} \leq \frac{r^2-1}{8r}$.

3. If r is even and $r > 2$, then

$$\frac{\overline{jb}(r-\overline{jb})}{2r} \le \begin{cases} \frac{\frac{r-2}{2}\frac{r+2}{2}}{2r} = \frac{r^2-4}{8r}, & \text{if } \overline{jb} \neq r/2; \\ \frac{r^2}{8r}, & \text{if } \overline{jb} = r/2. \end{cases}$$

Clearly, $b \neq r/2$ under the same situation. Since $\overline{jb} = r/2$ and $\overline{(j-1)b} = r/2$ can not hold simultaneously, we have

$$\frac{\overline{(j-1)b}(r-\overline{(j-1)b})}{2r} + \frac{\overline{jb}(r-\overline{jb})}{2r} \le \frac{r^2-4}{8r} + \frac{r^2}{8r} \le \frac{2\cdot(r^2-1)}{8r}.$$

Hence, when r is even and $r > 2$, we have

$$\sum_{j=1}^{n} \frac{\overline{jb}(r-\overline{jb})}{2r} \le n \cdot \frac{r^2-1}{8r}. \tag{4.3.1}$$

By the way, inequality (4.3.1) also holds when r is odd.

Recall that we have

$$\sum_{j=1}^{r} \frac{\overline{jb}(r-\overline{jb})}{2r} = \frac{r^2-1}{12}.$$

Hence, whenever $r > 2$ and $n \ge \frac{r_{\max}t}{3}$, we have

$$\begin{aligned} \sum_{j=1}^{n} \frac{\overline{jb}(r-\overline{jb})}{2r} &= \lfloor\frac{n}{r}\rfloor\frac{r^2-1}{12} + \sum_{j=1}^{\overline{n}} \frac{\overline{jb}(r-\overline{jb})}{2r} \\ &\le \lfloor\frac{n}{r}\rfloor\frac{r^2-1}{12} + \min\Big\{\overline{n}\cdot\frac{r^2-1}{8r}, \frac{r^2-1}{12}\Big\} \\ &\le \frac{r^2-1}{12r}(n+\frac{r}{3}) \\ &\le \frac{r^2-1}{12r}\cdot\frac{(t+1)n}{t}. \end{aligned} \tag{4.3.2}$$

We prove the second inequality here. Assume, to the contrary, that

$$\lfloor\frac{n}{r}\rfloor\frac{r^2-1}{12} + \overline{n}\cdot\frac{r^2-1}{8r} > \frac{r^2-1}{12r}(n+\frac{r}{3}), \tag{4.3.3}$$

and

$$\lfloor\frac{n}{r}\rfloor\frac{r^2-1}{12} + \frac{r^2-1}{12} > \frac{r^2-1}{12r}(n+\frac{r}{3}). \tag{4.3.4}$$

Inequality (4.3.3) implies $\overline{n} > \frac{2r}{3}$. But from inequality (4.3.4), we have $\overline{n} < \frac{2r}{3}$, a contradiction.

Since X is weak $\mathbb{Q}$-Fano, recall that we have inequality

$$\sum_i (r_i - \frac{1}{r_i}) \leq 24$$

by inequality (4.1.1). Denote by N_2 the number of $r_i = 2$ in B_X. Then, if $n \geq \frac{r_{\max} t}{3}$,

$$\begin{aligned}
l(-n) &= \sum_i \sum_{j=1}^{n} \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i} \\
&= \frac{N_2}{4}\lfloor \frac{n+1}{2} \rfloor + \sum_{r_i>2} \sum_{j=1}^{n} \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i} \\
&\leq \frac{N_2}{4}\lfloor \frac{n+1}{2} \rfloor + \frac{(t+1)n}{t} \sum_{r_i>2} \frac{r_i^2 - 1}{12 r_i} \\
&\leq \frac{N_2}{4}\lfloor \frac{n+1}{2} \rfloor + \frac{(t+1)n}{t} \cdot \frac{24 - \frac{3}{2}N_2}{12} \\
&\leq \frac{2(t+1)n}{t} - N_2\Big(\frac{(t+1)n}{8t} - \frac{1}{4}\lfloor \frac{n+1}{2} \rfloor\Big) \\
&\leq \frac{2(t+1)n}{t}
\end{aligned}$$

where $\frac{(t+1)n}{8t} - \frac{1}{4}\lfloor \frac{n+1}{2} \rfloor \geq 0$ whenever $n \geq t$. Hence

$$\begin{aligned}
P_{-n} &= \frac{1}{12} n(n+1)(2n+1)(-K_X^3) + 2n + 1 - l(-n) \\
&\geq \frac{1}{6} n^3(-K_X^3) + \frac{n^2}{4}(-K_X^3) + 1 - \frac{2n}{t}.
\end{aligned}$$

By [CC08], $-K_X^3 \geq \frac{1}{330}$. Hence $\frac{n^2}{4}(-K_X^3) \geq 1$ if $n \geq 37$. If $m \geq \sqrt{6r_X + \frac{12}{t(-K_X^3)}}$, then

$$\begin{aligned}
P_{-m} &\geq \frac{1}{6} m^3(-K_X^3) + 2 - \frac{2m}{t} \\
&\geq \frac{1}{6}\Big(6r_X + \frac{12}{t(-K_X^3)}\Big) m(-K_X^3) + 2 - \frac{2m}{t} \\
&= r_X(-K_X^3)m + 2.
\end{aligned}$$

We complete the proof. $\square$

In practice, we will take a suitable t to apply Proposition 4.3.2. Note that $r_{\max} \le 24$.

Proposition 4.3.3. *Let X be a weak $\mathbb{Q}$-Fano 3-fold.*

(i) If $r_X \le 660$, then $\sqrt{6r_X + \frac{3}{2(-K_X^3)}} < 67$. In particular,

$$P_{-m} \ge r_X(-K_X^3)m + 2$$

for $m \ge 67$.

(ii) If $r_X > 660$, then $r_X = 840$, and $P_{-m} \ge r_X(-K_X^3)m + 2$ for $m \ge 71$.

Proof. Statement (i) is clear since $-K_X^3 \ge \frac{1}{330}$ by [CC08] and take $t = 8$ in Proposition 4.3.2. We mainly prove (ii) here.

First of all, by Proposition 4.1.1, $r_X = 840$ and $\mathcal{R} = (3, 5, 7, 8)$ or $(2, 3, 5, 7, 8)$.

For $r > 2$, we use the inequality (4.3.2) (in the proof of Proposition 4.3.2) that

$$\sum_{j=1}^{n} \frac{\overline{jb}(r - \overline{jb})}{2r} \le \frac{r^2 - 1}{12r}(n + \frac{r}{3}).$$

Then

$$\begin{aligned}
l(-n) &= \sum_i \sum_{j=1}^{n} \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i} \\
&\le \frac{N_2}{4}\lfloor \frac{n+1}{2} \rfloor + \sum_{r_i > 2} \frac{r_i^2 - 1}{12r_i}(n + \frac{r_i}{3}) \\
&\le \frac{n+1}{8} + \frac{3^2 - 1}{12 \cdot 3}(n+1) + \frac{5^2 - 1}{12 \cdot 5}(n + \frac{5}{3}) \\
&\quad + \frac{7^2 - 1}{12 \cdot 7}(n + \frac{7}{3}) + \frac{8^2 - 1}{12 \cdot 8}(n + \frac{8}{3}) \\
&= \frac{19907n}{10080} + \frac{295}{72} \\
&\le 2n + \frac{7}{3}
\end{aligned}$$

as long as $n \ge 71$.

Hence

$$P_{-n} = \frac{1}{12}n(n+1)(2n+1)(-K_X^3) + 2n + 1 - l(-n)$$

$$\geq \frac{1}{6}n^3(-K_X^3) + \big(\frac{n^2}{4}(-K_X^3) - \frac{10}{3}\big) + 2.$$

By [CC08], $-K_X^3 \geq \frac{1}{330}$. Hence $\frac{n^2}{4}(-K_X^3) \geq \frac{10}{3}$ whenever $n \geq 71$. If $m \geq 71 > \sqrt{6r_X}$, then

$$\begin{aligned}P_{-m} &\geq \frac{1}{6}m^3(-K_X^3) + 2\\ &\geq \frac{1}{6}(6r_X)m(-K_X^3) + 2\\ &= r_X(-K_X^3)m + 2.\end{aligned}$$

We finish the proof. □

Theorem 1.2.14 directly follows from Corollary 4.3.1 and Proposition 4.3.3.

4.4 Birationality and generic finiteness

In this section, we consider the birationality and generic finiteness of anti-pluricanonical maps φ_{-m}.

4.4.1 Main reduction

Recall that by Lemma 2.7.2, we can reduce the birationality and generic finiteness problems on X to that on Y.

Lemma 4.4.1 (cf. [Chen11, 2.6]). *Let X be a weak $\mathbb{Q}$-Fano 3-fold and $\pi : Y \longrightarrow X$ a resolution. Then, for any $m > 0$, φ_{-m} is birational (resp. generically finite) if and only if so is $\Phi_{|K_Y + \lceil (m+1)\pi^*(-K_X)\rceil|}$.*

4.4.2 Key theorem

Let X be a weak $\mathbb{Q}$-Fano 3-fold on which $P_{-m_0} \geq 2$ for some integer $m_0 > 0$. Suppose that $m_1 \geq m_0$ is an integer with $P_{-m_1} \geq 2$ and that $|-m_1K_X|$ and $|-m_0K_X|$ are not composed with the same pencil. Recall that, for any $m > 0$ with $P_{-m} > 1$,

$$\iota(m) = \begin{cases} 1, & \text{if } |-mK_X| \text{ is not composed with a pencil;}\\ P_{-m} - 1, & \text{if } |-mK_X| \text{ is composed with a pencil.}\end{cases}$$

Set $D := -m_0K_X$ and keep the same notation as in Section 2.4. We may modify the resolution π in Section 2.4 such that the movable part $|M_{-m}|$

of $|\lfloor \pi^*(-mK_X)\rfloor|$ is base point free for all $m_0 \le m \le m_1$. Pick a generic irreducible element S of $|M_{-m_0}|$. We have

$$m_0\pi^*(-K_X) = \iota(m_0)S + F_{m_0}$$

for some effective $\mathbb{Q}$-divisor F_{m_0}. In particular, we see that

$$\frac{m_0}{\iota(m_0)}\pi^*(-K_X) - S \sim_{\mathbb{Q}} \text{effective } \mathbb{Q}\text{-divisor.}$$

Define the real number

$$\mu_0 = \mu_0(|S|) := \inf\{t \in \mathbb{Q}^+ \mid t\pi^*(-K_X) - S \sim_{\mathbb{Q}} \text{effective } \mathbb{Q}\text{-divisor}\}.$$

Remark 4.4.2. Clearly, we have $\mu_0 \le \frac{m_0}{\iota(m_0)}$. For all k such that $|-kK_X|$ and $|-m_0K_X|$ are composed with the same pencil, we have

$$k\pi^*(-K_X) = \iota(k)S + F_k$$

for some effective $\mathbb{Q}$-divisor F_k, and hence $\mu_0 \le \frac{k}{\iota(k)}$.

By our assumption on $|-m_1K_X|$, we know that $|G| = |M_{-m_1}|_S|$ is a base point free linear system on S and $h^0(S, G) \ge 2$. Denote by C a generic irreducible element of $|G|$. Since $m_1\pi^*(-K_X) \ge M_{-m_1}$, we have

$$m_1\pi^*(-K_X)|_S \equiv C + H$$

where H is an effective $\mathbb{Q}$-divisor on S.

We define two numbers which will be the key invariants accounting for the birationality and generic finiteness of φ_{-m}. They are

$$\begin{aligned} \zeta &:= (\pi^*(-K_X) \cdot C)_Y = (\pi^*(-K_X)|_S \cdot C)_S \text{ and} \\ \epsilon(m) &:= (m + 1 - \mu_0 - m_1)\zeta. \end{aligned}$$

Note that ζ and $\epsilon(m)$ are birational invariants by projection formula. Hence we can modify π if necessary.

While studying the birationality and generic finiteness of φ_{-m}, we always require that the linear system $\Lambda_m := |K_Y + \lceil (m+1)\pi^*(-K_X)\rceil|$ satisfies the following assumption for some integer $m > 0$.

Assumption 4.4.3. Keep the notation as above.

(1) The linear system Λ_m distinguishes different generic irreducible elements of $|M_{-m_0}|$ (namely, $\Phi_{\Lambda_m}(S') \ne \Phi_{\Lambda_m}(S'')$ for two different generic irreducible elements S', S'' of $|M_{-m_0}|$).

(2) The linear system $\Lambda_m|_S$ distinguishes different generic irreducible elements of the linear system $|G| = |M_{-m_1}|_S|$ on S.

The following is our key theorem.

Theorem 4.4.4 (cf. [Chen11, Theorem 3.5])**.** *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Keep the notation as above. Let $m > 0$ be an integer. If Assumption 4.4.3 is satisfied and $\epsilon(m) > 2$ (resp. $\epsilon(m) > \max\{2 - g(C), 0\}$), then φ_{-m} is birational (resp. generically finite) onto its image.*

Proof. By Lemma 4.4.1, we only need to prove the birationality (resp. generic finiteness) of Φ_{Λ_m}. Since Assumption 4.4.3(1) is satisfied, the usual birationality principle reduces the birationality (resp. generic finiteness) of Φ_{Λ_m} to that of $\Phi_{\Lambda_m}|_S$ for a generic irreducible element S of $|M_{-m_0}|$. Similarly, due to Assumption 4.4.3(2), we only need to prove the birationality (resp. generic finiteness) of $\Phi_{\Lambda_m}|_C$ for a generic irreducible element C of $|G|$. Now we show how to restrict the linear system Λ_m to C.

Now assume $\epsilon(m) > 0$. We can find a sufficiently large integer n so that there exists a number $\mu_0^{(n)} \in \mathbb{Q}^+$ with $0 \le \mu_0^{(n)} - \mu_0 \le \frac{1}{n}$, $\lceil \epsilon(m, n) \rceil = \lceil \epsilon(m) \rceil$ where $\epsilon(m, n) := (m + 1 - \mu_0^{(n)} - m_1)\zeta$, and

$$\mu_0^{(n)} \pi^*(-K_X) \sim_{\mathbb{Q}} S + E^{(n)}$$

for an effective $\mathbb{Q}$-divisor $E^{(n)}$. In particular, $\epsilon(m, n) > 0$. Re-modify our π in Section 2.4 so that $E^{(n)}$ has simple normal crossing support.

For the given integer $m > 0$, we have

$$|K_Y + \lceil (m+1)\pi^*(-K_X) - E^{(n)} \rceil| \preceq |K_Y + \lceil (m+1)\pi^*(-K_X) \rceil|. \quad (4.4.1)$$

Since $\epsilon(m, n) > 0$, the $\mathbb{Q}$-divisor

$$(m+1)\pi^*(-K_X) - E^{(n)} - S \equiv (m + 1 - \mu_0^{(n)})\pi^*(-K_X)$$

is nef and big and thus

$$H^1(Y, K_Y + \lceil (m+1)\pi^*(-K_X) - E^{(n)} \rceil - S) = 0$$

by Kawamata–Viehweg vanishing theorem. Hence we have surjective map

$$H^0(Y, K_Y + \lceil (m+1)\pi^*(-K_X) - E^{(n)} \rceil) \longrightarrow H^0(S, K_S + L_{m,n}) \quad (4.4.2)$$

where

$$L_{m,n} := (\lceil (m+1)\pi^*(-K_X) - E^{(n)} \rceil - S)|_S \ge \lceil \mathcal{L}_{m,n} \rceil \quad (4.4.3)$$

and $\mathcal{L}_{m,n} := ((m+1)\pi^*(-K_X) - E^{(n)} - S)|_S$. Moreover, we have

$$m_1\pi^*(-K_X)|_S \equiv C + H$$

for an effective $\mathbb{Q}$-divisor H on S by the setting. Thus the $\mathbb{Q}$-divisor

$$\mathcal{L}_{m,n} - H - C \equiv (m + 1 - \mu_0^{(n)} - m_1)\pi^*(-K_X)|_S$$

is nef and big by $\epsilon(m,n) > 0$. And by Kawamata–Viehweg vanishing theorem again,

$$H^1(S, K_S + \lceil \mathcal{L}_{m,n} - H \rceil - C) = 0.$$

Therefore, we have surjective map

$$H^0(S, K_S + \lceil \mathcal{L}_{m,n} - H \rceil) \longrightarrow H^0(C, K_C + D_{m,n}) \tag{4.4.4}$$

where

$$D_{m,n} := \lceil \mathcal{L}_{m,n} - H - C \rceil|_C \geq \lceil \mathcal{D}_{m,n} \rceil \tag{4.4.5}$$

and $\mathcal{D}_{m,n} := (\mathcal{L}_{m,n} - H - C)|_C$ with $\deg\lceil \mathcal{D}_{m,n} \rceil \geq \lceil \epsilon(m,n) \rceil$.

Now by inequalities (4.4.1), (4.4.3), (4.4.5), and surjective maps (4.4.2), (4.4.4), to prove the birationality (resp. generic finiteness) of $\Phi_{\Lambda_m}|_C$, it is sufficient to prove that $|K_C + \lceil \mathcal{D}_{m,n} \rceil|$ gives a birational (resp. generically finite) map. Clearly this is the case whenever $\epsilon(m) > 2$ (resp. $> 2 - g(C)$), which in fact implies $\deg(\lceil \mathcal{D}_{m,n} \rceil) \geq \lceil \epsilon(m,n) \rceil \geq 3$ (resp. $\geq 3 - g(C)$) and $K_C + \lceil \mathcal{D}_{m,n} \rceil$ is very ample (resp. defines a finite map). We complete the proof. □

Corollary 4.4.5. *Keep the same notation as above. For any integer $m > 0$, set*

$$\epsilon(m,0) := (m + 1 - \frac{m_0}{\iota(m_0)} - m_1)\zeta.$$

If $\epsilon(m,0) > 0$, then

$$\Lambda_m|_S \succeq |K_S + L_m|$$

where $L_m := (\lceil (m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)} F_{m_0} \rceil - S)|_S$.

Proof. First of all, relation (4.4.1) reads

$$|K_Y + \lceil (m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)} F_{m_0} \rceil| \preceq |K_Y + \lceil (m+1)\pi^*(-K_X) \rceil|. \tag{4.4.6}$$

In fact, as long as $\epsilon(m,0)>0$, the front part of the proof of Theorem 4.4.4 is valid. In explicit, subjective map (4.4.2) reads the following surjective map

$$H^0(Y, K_Y + \lceil (m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)}F_{m_0}\rceil) \longrightarrow H^0(S, K_S + L_m) \quad (4.4.7)$$

where

$$L_m := (\lceil (m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)}F_{m_0}\rceil - S)|_S. \qquad (4.4.8)$$

Hence we have proved the statement. $\square$

4.4.3 Applications

In order to apply Theorem 4.4.4, we need to verify Assumption 4.4.3 and lower bound of $\epsilon(m)$ in advance, for which one of the crucial steps is to estimate the lower bound of ζ.

Proposition 4.4.6 (cf. [Chen11, Theorem 3.2])**.** *Let $m>0$ be an integer. Keep the same notation as in Subsection 4.4.2.*

(i) If $g(C)>0$ and $\epsilon(m)>1$, then $\zeta \geq \frac{2g(C)-2+\lceil \epsilon(m)\rceil}{m}$;

(ii) Moreover, if $g(C)>0$, then

$$\zeta \geq \frac{2g(C)-1}{\mu_0+m_1};$$

(iii) If $g(C)=1$, then $\zeta \geq \frac{1}{r_{\max}}$, where $r_{\max} = \max\{r_i \in B_X\}$ is the maximum of local indices of singularities;

(iv) If $g(C)=0$, then $\zeta \geq 2$;

(v) If $h^0(-\nu K_X)>0$ for some integer ν, then $\zeta \geq \frac{1}{\nu r_{\max}}$.

Proof. (i). In the proof of Theorem 4.4.4, if $g(C)>0$ and $\epsilon(m)>1$ then $|K_C + \lceil \mathcal{D}_{m,n}\rceil|$ is base point free with

$$\deg(K_C + \lceil \mathcal{D}_{m,n}\rceil) \geq 2g(C) - 2 + \lceil \epsilon(m,n)\rceil = 2g(C) - 2 + \lceil \epsilon(m)\rceil.$$

Denote by $\mathcal{N}_m$ the movable part of $|K_S + \lceil \mathcal{L}_{m,n} - H\rceil|$. Recall that M_{-m} is the movable part of $\lfloor m\pi^*(-K_X)\rfloor$ and

$$H^0(\lfloor m\pi^*(-K_X)\rfloor) = H^0(K_Y + \lceil (m+1)\pi^*(-K_X)\rceil).$$

Noting the relations (4.4.1), (4.4.2), and $|K_S + \lceil \mathcal{L}_{m,n} - H \rceil| \preceq |K_S + \lceil \mathcal{L}_{m,n} \rceil|$ while applying [Chen01, Lemma 2.7], we get

$$m\pi^*(-K_X)|_S \geq M_{-m}|_S \geq \mathcal{N}_m$$

and $\mathcal{N}_m|_C \geq K_C + \lceil \mathcal{D}_{m,n} \rceil$ since the latter one is base point free. So we have

$$m\zeta = m\pi^*(-K_X)|_S \cdot C \geq \mathcal{N}_m \cdot C \geq \deg(K_C + \lceil \mathcal{D}_{m,n} \rceil).$$

Hence

$$m\zeta \geq 2g(C) - 2 + \lceil \epsilon(m) \rceil.$$

(ii). Take $m' = \min\{m \mid \epsilon(m) > 1\}$, then (i) implies $\zeta \geq \frac{2g(C)}{m'}$. We may assume that $m' > \mu_0 + m_1$ otherwise $\zeta \geq \frac{2g(C)}{\mu_0 + m_1}$. Hence

$$\begin{aligned}\epsilon(m' - 1) &= (m' - 1 + 1 - \mu_0 - m_1)\zeta \\ &\geq (m' - \mu_0 - m_1)\frac{2g(C)}{m'}.\end{aligned}$$

By the minimality of m', it follows that $\epsilon(m' - 1) \leq 1$. Hence $m' \leq \frac{2g(C)}{2g(C)-1}(\mu_0 + m_1)$. Then

$$\zeta \geq \frac{2g(C)}{m'} \geq \frac{2g(C) - 1}{\mu_0 + m_1}.$$

(iii). If $g(C) = 1$, then

$$\begin{aligned}\zeta &= (\pi^*(-K_X) \cdot C)_Y = ((-K_Y + E_\pi) \cdot C)_Y \\ &= (-(K_Y + S) \cdot C + S \cdot C + E_\pi \cdot C)_Y \\ &= (-K_S \cdot C)_S + (S \cdot C + E_\pi \cdot C)_Y \\ &= (C^2)_S + (S \cdot C + E_\pi \cdot C)_Y.\end{aligned}$$

Since C is free on the smooth surface S, $(C^2)_S$, $(S \cdot C)_Y$, and $(E_\pi \cdot C)_Y$ are non-negative. Since $(C^2)_S$ and $(S \cdot C)_Y$ are integers, we may assume $(C^2)_S = (S \cdot C)_Y = 0$ otherwise $\zeta \geq 1$. Hence $\zeta = E_\pi \cdot C$.

On the other hand, take $q : W \to X$ is the resolution of isolated singularities and we may assume that Y dominates W by $p : Y \to W$. Then we write

$$K_W = q^* K_X + \Delta.$$

Here

$$\Delta = \sum \frac{a_i}{r_i} E_i$$

where E_i is the exceptional divisor over an isolated singular point of index r_i for some $r_i \in B_X$ and a_i is a positive integer. Then

$$E_\pi = K_Y - p^*K_W + p^*\Delta.$$

Take $r_{\max} = \max\{r_i\}$. Then all the coefficients of E_π are larger than $\frac{1}{r_{\max}}$ since $K_Y - p^*K_W$ is integral and effective and $\mathrm{Supp}(E_\pi) = \mathrm{Supp}(K_Y - p^*K_W + p_*^{-1}\Delta)$. By $E_\pi \cdot C = \zeta > 0$, we know that there is at least one component E of E_π such that $E \cdot C > 0$. Then $E_\pi \cdot C \geq \frac{1}{r_{\max}} E \cdot C \geq \frac{1}{r_{\max}}$.

(iv). If $g(C) = 0$, then

$$\begin{aligned} \zeta &= (\pi^*(-K_X)|_S \cdot C)_S = ((-K_Y + E_\pi)|_S \cdot C)_S \\ &\geq (-K_Y|_S \cdot C)_S \geq (-K_S \cdot C)_S \geq -\deg(K_C) = 2. \end{aligned}$$

(v). If $h^0(-\nu K_X) > 0$ for some integer ν, then $-\nu K_X \sim D$ for some effective Weil divisor D. Similarly as (iii), π^*D is an effective $\mathbb{Q}$-divisor with all the coefficients larger than $\frac{1}{r_{\max}}$. By $\pi^*D \cdot C = \nu\zeta > 0$, we know that there is at least one component D_1 of π^*D such that $D_1 \cdot C > 0$. Then $\zeta = \frac{1}{\nu}\pi^*D \cdot C \geq \frac{1}{\nu r_{\max}} D_1 \cdot C \geq \frac{1}{\nu r_{\max}}$. $\square$

To verify Assumption 4.4.3(1), we have the following proposition.

Proposition 4.4.7 (cf. [Chen11, Proposition 3.6]). *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Keep the same notation as Subsection 4.4.2. Then Assumption 4.4.3(1) is satisfied for all*

$$m \geq \begin{cases} m_0 + 6, & \text{if } m_0 \geq 2; \\ 2, & \text{if } m_0 = 1. \end{cases}$$

Proof. We have

$$\begin{aligned} &K_Y + \lceil (m+1)\pi^*(-K_X) \rceil \\ \geq\ & K_Y + \lceil (m - m_0 + 1)\pi^*(-K_X) + M_{-m_0} \rceil \\ =\ & (K_Y + \lceil (m - m_0 + 1)\pi^*(-K_X) \rceil) + M_{-m_0} \\ \geq\ & M_{-m_0}. \end{aligned}$$

The last inequality is due to

$$h^0(K_Y + \lceil (m - m_0 + 1)\pi^*(-K_X) \rceil) = h^0(-(m - m_0)K_X) > 0$$

by Lemma 4.4.1 and [Chen11, Appendix], since $m - m_0 \geq 6$ whenever $m_0 \geq 2$ (resp. ≥ 1 whenever $m_0 = 1$).

When $f : Y \to \Gamma$ is of type (f_{np}), [Tan71, Lemma 2] implies that Λ_m can distinguish different generic irreducible elements of $|M_{-m_0}|$. When f is of type (f_{p}), since the rational (i.e. $\Gamma \cong \mathbb{P}^1$) pencil $|M_{-m_0}|$ can already separate different fibers of f, Λ_m can naturally distinguish different generic irreducible elements of $|M_{-m_0}|$. $\square$

It is slightly more complicated to verify Assumption 4.4.3(2).

Lemma 4.4.8 (cf. [Chen11, Lemma 3.7])**.** *Let T be a smooth projective surface with a base point free linear system $|G|$. Let Q be an arbitrary $\mathbb{Q}$-divisor on T. Denote by C a generic irreducible element of $|G|$. Then the linear system $|K_T + \lceil Q \rceil + G|$ can distinguish different generic irreducible elements of $|G|$ under one of the following conditions:*

(i) $|G|$ is not composed with an irrational pencil of curves and $K_T + \lceil Q \rceil$ is effective;

(ii) $|G|$ is composed with an irrational pencil of curves, $g(C) > 0$, and Q is nef and big;

(iii) $|G|$ is composed with an irrational pencil of curves, $g(C) = 0$, Q is nef and big, and $Q \cdot G > 1$.

Proof. The statement corresponding to (i) follows from [Tan71, Lemma 2] and the fact that a rational pencil can automatically separate its different generic irreducible elements.

For situations (ii) and (iii), we pick a generic irreducible element C of $|G|$. Then, since $h^0(S, G) \geq 2$, $G \equiv sC$ for some integer $s \geq 2$ and $C^2 = 0$. Denote by C_1 and C_2 two generic irreducible elements of $|G|$ such that $C_1 + C_2 \leq |G|$. Then Kawamata–Viehweg vanishing theorem gives the surjective map

$$H^0(T, K_T + \lceil Q \rceil + G) \longrightarrow H^0(C, K_{C_1} + D_1) \oplus H^0(C_2, K_{C_2} + D_2)$$

where $D_i := (\lceil Q \rceil + G - C_i)|_{C_i}$ with $\deg(D_i) \geq Q \cdot C_i > 0$ for $i = 1, 2$.

If $g(C) > 0$, Riemann–Roch formula gives $h^0(C_i, K_{C_i} + D_i) > 0$ for $i = 1, 2$. Thus $|K_T + \lceil Q \rceil + G|$ can distinguish C_1 and C_2.

If $g(C) = 0$ and $Q \cdot C > 1$, then $h^0(C_i, K_{C_i} + D_i) > 0$ for $i = 1, 2$. So $|K_T + \lceil Q \rceil + G|$ can also distinguish C_1 and C_2. $\square$

Proposition 4.4.9 (cf. [Chen11, Proposition 3.8, 3.9])**.** *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Keep the same notation as in Subsection 4.4.2. Then Assumption 4.4.3(2) is satisfied for all*

$$m \geq \begin{cases} m_0 + m_1 + 6, & \text{if } m_0 \geq 2; \\ m_1 + 2, & \text{if } m_0 = 1. \end{cases}$$

Proof. Assuming $m \geq m_0 + m_1$, we have $\epsilon(m, 0) > 0$, and Corollary 4.4.5 implies that

$$\Lambda_m|_S \succeq |K_S + L_m|.$$

It suffices to prove that $|K_S + L_m|$ can distinguish different generic irreducible elements of $|G|$.

For a suitable integer $m > 0$, we have the following relations:

$$\begin{aligned} K_S + L_m &= (K_Y)|_S + \lceil (m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)} F_{m_0} \rceil|_S \\ &\geq (K_Y + \lceil (m+1-m_0-m_1)\pi^*(-K_X) \rceil)|_S + M_{-m_1}|_S. \end{aligned}$$

Thus, if $|G|$ is not composed with an irrational pencil of curves, $|K_S + L_m|$ can distinguish different irreducible elements provided that

$$K_Y + \lceil (m+1-m_0-m_1)\pi^*(-K_X) \rceil$$

is effective, which holds for $m - m_0 - m_1 \geq 6$ whenever $m_0 \geq 2$ (resp. ≥ 1 whenever $m_0 = 1$) by [Chen11, Appendix].

Assume $|G|$ is composed with an irrational pencil of curves. we have

$$\begin{aligned} K_S + L_m &\geq K_S + \lceil \mathcal{L}_m \rceil \\ &= K_S + \lceil ((m+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)} F_{m_0} - S)|_S \rceil \\ &\geq K_S + \lceil ((m-m_1+1)\pi^*(-K_X) - \frac{1}{\iota(m_0)} F_{m_0} - S)|_S \rceil + M_{-m_1}|_S. \end{aligned}$$

We can take $Q = ((m - m_1 + 1)\pi^*(-K_X) - \frac{1}{\iota(m_0)} F_{m_0} - S)|_S$ in Lemma 4.4.8 since $\epsilon(m, 0) > 0$.

If $g(C) > 0$, Lemma 4.4.8(ii) implies that Assumption 4.4.3(2) is satisfied for $m \geq m_0 + m_1$.

If $g(C) = 0$, by Lemma 4.4.8(iii), we need the condition $\epsilon(m, 0) = (m + 1 - \frac{m_0}{\iota} - m_1)\zeta = Q \cdot C > 1$. But this holds automatically for $m \geq m_0 + m_1$ by Proposition 4.4.6(iv).

We complete the proof. □

Now we can treat the birationality of φ_{-m} using Theorem 4.4.4.

Theorem 4.4.10 (cf. [Chen11, Theorem 4.1, 4.2, 4.5])**.** *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Let ν_0 be an integer such that $h^0(-\nu_0 K_X) > 0$. Keep the same notation as in Subsection 4.4.2. Then φ_{-m} is birational onto its image if one of the following holds:*

(i) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor 3\mu_0 \rfloor + 3m_1\}$*;*

(ii) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor \frac{5}{3}\mu_0 + \frac{5}{3}m_1 \rfloor, \lfloor \mu_0 \rfloor + m_1 + 2r_{\max}\}$*;*

(iii) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor \mu_0 \rfloor + m_1 + 2\nu_0 r_{\max}\}$*,*

where $a(m_0) = \begin{cases} 6, & \text{if } m_0 \geq 2; \\ 1, & \text{if } m_0 = 1. \end{cases}$

Proof. By Propositions 4.4.7 and 4.4.9, Assumption 4.4.3 is satisfied if $m \geq m_0 + m_1 + a(m_0)$.

By Proposition 4.4.6(v), $\zeta \geq \frac{1}{\nu_0 r_{\max}}$. If $m \geq \lfloor \mu_0 \rfloor + m_1 + 2\nu_0 r_{\max}$, then $\epsilon(m) = (m + 1 - \mu_0 - m_1)\zeta > 2$, which implies (iii).

For (i) and (ii), we will discuss on the value of $g(C)$.

Case 1. $g(C) = 0$.

By Proposition 4.4.6(iv), $\zeta \geq 2$. If $m \geq \lfloor \mu_0 \rfloor + m_1 + 1$, then $\epsilon(m) = (m + 1 - \mu_0 - m_1)\zeta > 2$.

Case 2. $g(C) \geq 2$.

By Proposition 4.4.6(ii), $\zeta \geq \frac{3}{\mu_0 + m_1}$. If $m \geq \lfloor \frac{5}{3}\mu_0 + \frac{5}{3}m_1 \rfloor$ then $\epsilon(m) \geq (m + 1 - \mu_0 - m_1)\zeta > 2$.

Case 3. $g(C) = 1$.

By Proposition 4.4.6(ii), $\zeta \geq \frac{1}{\mu_0 + m_1}$. If $m \geq \lfloor 3\mu_0 \rfloor + 3m_1$, then $\epsilon(m) = (m + 1 - \mu_0 - m_1)\zeta > 2$. So we have proved (i). On the other hand, by Proposition 4.4.6(iii), $\zeta \geq \frac{1}{r_{\max}}$. If $m \geq \lfloor \mu_0 \rfloor + m_1 + 2r_{\max}$, then $\epsilon(m) = (m + 1 - \mu_0 - m_1)\zeta > 2$. Thus (ii) is proved. $\square$

Similarly, we have a generic finiteness criterion.

Theorem 4.4.11. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Keep the same notation as in Subsection 4.4.2. Then φ_{-m} is generically finite onto its image if one of the following holds:*

(i) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor 2\mu_0 \rfloor + 2m_1\}$*;*

(ii) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor \mu_0 \rfloor + m_1 + r_{\max}\}$*,*

where $a(m_0) = \begin{cases} 6, & \text{if } m_0 \geq 2; \\ 1, & \text{if } m_0 = 1. \end{cases}$

Proof. By Propositions 4.4.7 and 4.4.9, Assumption 4.4.3 is satisfied if $m \geq m_0 + m_1 + a(m_0)$.

We will discuss on the value of $g(C)$.

Case 1. $g(C) = 0$.

By Proposition 4.4.6(iv), $\zeta \geq 2$. If $m \geq \lfloor \mu_0 \rfloor + m_1 + 1$, then $\epsilon(m) = (m + 1 - \mu_0 - m_1)\zeta > 2$.

Case 2. $g(C) \geq 2$.

If $m \geq \lfloor \mu_0 \rfloor + m_1$ then $\epsilon(m) \geq (m + 1 - \mu_0 - m_1)\zeta > 0$.

Case 3. $g(C) = 1$.

By Proposition 4.4.6(ii), $\zeta \geq \frac{1}{\mu_0 + m_1}$. If $m \geq \lfloor 2\mu_0 \rfloor + 2m_1$, then $\epsilon(m) = (m + 1 - \mu_0 - m_1)\zeta > 1$. So we have proved (i). On the other hand, by Proposition 4.4.6(iii), $\zeta \geq \frac{1}{r_{\max}}$. If $m \geq \lfloor \mu_0 \rfloor + m_1 + r_{\max}$, then $\epsilon(m) = (m + 1 - \mu_0 - m_1)\zeta > 1$. Thus (ii) is proved. □

In practice, usually we just use the fact $\mu_0 \leq \frac{m_0}{\iota(m_0)} \leq m_0$. For very few cases, we will utilize a precise upper bound of μ_0 rather than m_0 by Remark 4.4.2.

Theorems 4.4.10 and 4.4.11 are optimal in some cases due to the following examples.

Example 4.4.12 ([IF00, List 16.6])**.** Consider general weighted hypersurface $X_{6d} \subset \mathbb{P}(1, a, b, 2d, 3d)$ where $1 \leq a \leq b$ and $d = a + b$ such that X_{6d} is a $\mathbb{Q}$-Fano 3-fold with $r_{\max} = d$. By [IF00, List 16.6], there are exactly 12 such examples. Then φ_{-3d} is birational onto its image but $\varphi_{-(3d-1)}$ is not, and φ_{-2d} is generically finite onto its image but $\varphi_{-(2d-1)}$ is not.

On the other hand, We can take $\nu_0 = 1$, $m_0 = \mu_0 = a$ and $m_1 = b$, then

$$\begin{aligned} 3d &= \lfloor 3\mu_0 \rfloor + 3m_1 \\ &= \lfloor \mu_0 \rfloor + m_1 + 2r_{\max} \\ &= \lfloor \mu_0 \rfloor + m_1 + 2\nu_0 r_{\max} \end{aligned}$$

and

$$\begin{aligned} 2d &= \lfloor 2\mu_0 \rfloor + 2m_1 \\ &= \lfloor \mu_0 \rfloor + m_1 + r_{\max}. \end{aligned}$$

Hence Theorems 4.4.10 and 4.4.11 tell that φ_{-m} is birational onto its image for $m \geq 3d$, and φ_{-m} is generically finite onto its image for $m \geq 2d$.

Theorems 4.4.10 and 4.4.11 directly imply the following result which generalizes a result of Fukuda [Fuk91, Main theorem].

Corollary 4.4.13. *Let X be a weak $\mathbb{Q}$-Fano 3-fold with Gorenstein singularities. Then φ_{-m} is birational (resp. generically finite) onto its image for all $m \geq 4$ (resp. ≥ 3).*

Proof. By Reid's formula, $P_{-1} = \frac{1}{2}(-K_X^3) + 3 > 3$. Hence we can take $m_0 = \nu_0 = 1$.

If $|-K_X|$ is not composed with a pencil, then we can take $m_1 = 1$ and $\mu_0 \leq m_0 = 1$. And the result follows directly from Theorems 4.4.10(iii) and 4.4.11(ii).

If $|-K_X|$ is composed with a pencil, then $\mu_0 \leq \frac{m_0}{\iota(m_0)} < \frac{1}{2}$. By Reid's formula again, $P_{-2} = \frac{5}{2}(-K_X^3) + 5 > r_X(-K_X^3)2 + 1$. We can take $m_1 = 2$ by Corollary 4.3.1. And the result follows directly from Theorems 4.4.10(iii) and 4.4.11(ii). □

4.4.4 Proof of Theorems 1.2.11 and 1.2.15

Now we prove our main results on the birationality of φ_{-m}.

Proof of Theorem 1.2.11. To apply Theorem 4.4.10, we always use the fact $\mu_0 \leq m_0$. By [CC08, Theorem 1.1] and Theorem 1.2.9, we can take $m_0 \leq 8$ and $m_1 \leq 10$ to apply Theorem 4.4.10(i) and (ii). Hence $m_0 + m_1 + 6 \leq 24$ and $\frac{5}{3}(m_0 + m_1) \leq 30$. By Theorem 4.4.10, it is sufficient to prove that either $3m_0 + 3m_1 \leq 39$ or $m_0 + m_1 + 2r_{\max} \leq 39$ holds if we choose suitable m_0 and m_1. (Note that ν_0 is not used in this proof.)

Case 1. $P_{-1} \geq 2$.

In this case, we can take $m_0 = 1$ and $m_1 \leq 6$ (resp. $m_1 = 1$ whenever $P_{-1} > 2$) by Theorem 4.2.8. Hence $3m_0 + 3m_1 \leq 21$ (resp. ≤ 6 whenever $P_{-1} > 2$). This proves Corollary 1.2.12.

Case 2. $P_{-1} = 1$.

Recall the proof of Theorem 4.2.10. We take $m_0 = n_0$. If $m_0 \leq 5$, then we can take $m_1 \leq 7$ and hence $3m_0 + 3m_1 \leq 36$. Similarly, if $m_0 = 6$ and if we can take $m_1 \leq 7$, then $3m_0 + 3m_1 \leq 39$.

If $m_0 = n_0 = 6$ and $\delta_1(X) = 8$, we can take $m_1 = 8$. Theorem 4.2.4 implies that

$$P_{-1} = P_{-2} = P_{-3} = P_{-4} = P_{-5} = 1, P_{-6} = P_{-7} = 2.$$

Then $n^0_{1,2} = 2$, $n^0_{1,3} = 2$, $n^0_{1,4} = 2 - \sigma_5$, $\epsilon_5 = 2 - \sigma_5$, $0 = \epsilon_6 = 3 - \epsilon$. Hence $\epsilon = 3$ and $\sigma_5 \leq 2$, and this implies $(\sigma_5, n^0_{1,5}) = (2, 1)$. Then $\epsilon_5 = 0$ and $B^{(5)}(B) = \{2 \times (1,2), 2 \times (1,3), (1,5), (1,s)\}$ for some $s \geq 6$. This implies $\epsilon_7 = 0$ since there are no further packings. On the other hand, $\epsilon_7 = 2 - 2\sigma_5 + 2n^0_{1,5} + n^0_{1,6}$. Hence $n^0_{1,6} = 0$ and $B^{(7)} = \{2 \times (1,2), 2 \times (1,3), (1,5), (1,s)\}$ with $s \geq 7$. Since $B^{(7)}$ admits no prime packings, $B = B^{(7)}$. By inequalities (4.2.3) and (4.2.4), s can only be $8, 9, 10$. Hence $m_0 + m_1 + 2r_{\max} \leq 6 + 8 + 2 \times 10 = 34$.

If $m_0 = n_0 \geq 7$, then

$$P_{-1} = P_{-2} = P_{-3} = P_{-4} = P_{-5} = P_{-6} = 1.$$

The proof of Theorem 4.2.10 implies $B^{(5)} = \{(1,2), (2,5), (1,3), (1,4), (1,s)\}$ with $s \geq 6$. Since $\gamma(B^{(5)}) > 0$, we have $s \leq 11$. Noting that B is dominated by $B^{(5)}$, we see $r_{\max} \leq 11$. By Theorem 4.2.10, we can take $m_0 \leq 8$ and $m_1 \leq 9$. Hence $m_0 + m_1 + 2r_{\max} \leq 8 + 9 + 2 \times 11 = 39$.

Case 3. $P_{-1} = P_{-2} = 0$.

By the proof of Theorem 4.2.12 and Theorem 4.2.15, if B is of type No.1, No.2 or No.4, then we have $r_{\max} \leq 10$ and may take $m_0 = 8$, $m_1 = 10$. Hence $m_0 + m_1 + 2r_{\max} \leq 8 + 10 + 2 \times 10 = 38$. If B is of type No.5-No.6, then we have $r_{\max} \leq 7$ and may take $m_0 = 7$, $m_1 = 8$. Hence $m_0 + m_1 + 2r_{\max} \leq 7 + 8 + 2 \times 7 = 29$. If B is of type No.7-No.23, then we can take $m_0 = m_1 = 6$. Hence $3m_0 + 3m_1 \leq 36$. Now the remaining case is type No.3:

$$\{5 \times (1,2), 2 \times (1,3), (3,11)\}.$$

Recall that $P_{-8} = P_{-9} = 2$ and $-4K_X \sim E$ is a prime divisor by the proof of Theorem 4.2.15(i). By the proof of Theorem 4.2.4, $|-8K_X|$ has no fixed part. If $|-8K_X|$ and $|-9K_X|$ are composed with a same pencil, we can write

$$\begin{aligned} |-8K_X| &= |S'|, \\ |-9K_X| &= |S'| + F, \end{aligned}$$

where F is the fixed part. This implies that

$$-K_X \sim -9K_X - (-8K_X) = F,$$

which contradicts $P_{-1} = 0$. Hence $|-8K_X|$ and $|-9K_X|$ are composed with different pencils, and we can take $m_0 = 8$, $m_1 = 9$ and $m_0 + m_1 + 2r_{\max} = 39$.

Case 4. $P_{-1} = 0, P_{-2} > 0$.

By [CC08, Proposition 3.10, Case 1], we can take $m_0 = 6$. We can take m_1 the same as in the proof of Theorem 4.2.12 and Theorem 4.2.15. If $m_1 \leq 6$, then $3m_0 + 3m_1 \leq 36$. If $m_1 \geq 7$, observing Subsubcase II-3-ii and Subsubcase II-3-iii in the proof of Theorem 4.2.12, we can see that $r_{\max} \leq 11$ holds for any such basket except

$$B_d = \{4 \times (1,2), (6,13), (1,5)\}.$$

Except for B_d, we have $m_0 + m_1 + 2r_{\max} \leq 6 + 8 + 2 \times 11 = 36$. Now we deal with B_d. We claim that we can take $m_1 = 7$. Recall that

$$P_{-1} = P_{-3} = 0, P_{-2} = P_{-4} = P_{-5} = 1, P_{-6} = P_{-7} = 2.$$

Clearly $|-6K_X|$ and $|-7K_X|$ are both composed with pencils. We only need to show that they are composed with different pencils. To the contrary, we assume that $|-6K_X|$ and $|-7K_X|$ are composed with the same pencil. If $-2K_X \sim D$ is a prime divisor, then by the proof of Theorem 4.2.4, $|-6K_X|$ has no fixed part. By assumption, we can write

$$\begin{aligned} |-6K_X| &= |S'|, \\ |-7K_X| &= |S'| + F, \end{aligned}$$

where F is the fixed part. This implies that

$$-K_X \sim -7K_X - (-6K_X) = F,$$

a contradiction. Hence $-2K_X \sim D$ is not a prime divisor. By the proof of Theorem 4.2.15(ii), $D = E_1 + E_2$ with E_1 and E_2 different prime divisors. Also we can write

$$\begin{aligned} |-6K_X| &= |S'| + a_6E_1, \\ |-7K_X| &= |S'| + F, \end{aligned}$$

where a_6E_1 and F are the fixed parts with $a_6 \leq 3$. If $a_6 \leq 1$, then

$$S' \sim 3(E_1 + E_2) - a_6E_1 \geq 2E_1 + 2E_2 \sim -4K_X.$$

This implies $|-7K_X| \succeq |-4K_X|$, which contradicts $P_{-3} = 0$. If $a_6 = 3$, as in the proof of Theorem 4.2.2, take $m = 6$ and $E = E_1$ or $2E_1$ or $3E_1$, inequality (4.2.1) must fail for some singularity P in B_d. Clearly, such an offending singularity P must be "$(6, 13)$". By equality (4.2.2), the local index $i_P(E)$ of E can only be 9 or 11 since inequality (4.2.1) holds for other $0 \leq i \leq 12$ and $(b, r) = (6, 13)$. But clearly the local index $i_P(E_1)$, $i_P(2E_1)$, and $i_P(3E_1)$ can not be in the set $\{9, 11\}$ simultaneously, a contradiction. Finally we consider the case $a_6 = 2$. Write $-5K_X \sim B$ a fixed divisor. Then

$$B + S' + 2E_1 \sim -5K_X - 6K_X \sim -4K_X - 7K_X \sim 2E_1 + 2E_2 + S' + F,$$

that is, $B \sim 2E_2 + F$. Obviously, $F \neq 0$. As in the proof of Theorem 4.2.2, take $m = 5$ and $E = E_1$ or $2E_1$, inequality (4.2.1) must fail for some singularity P in B_d. Clearly, such an offending singularity P must be "$(6, 13)$".

By equality (4.2.2), the local index $i_P(E)$ of E can only be 10 or 11 since inequality (4.2.1) holds for other $0 \leq i \leq 12$ and $(b, r) = (6, 13)$. But clearly the local index $i_P(E_1)$, $i_P(2E_1)$ can not be in the set $\{10, 11\}$ simultaneously, a contradiction.

We complete the proof. $\square$

Note that Theorem 1.2.13 follows from the proof of Theorem 1.2.11 and Theorem 4.4.11.

Proof of Theorem 1.2.15. We shall apply Theorem 4.4.10 to treat arbitrary weak $\mathbb{Q}$-Fano 3-folds. We will choose suitable m_0 and m_1. Unless otherwise specified, we will use the fact $\mu_0 \leq m_0$.

Case I. $P_{-2} = 0$.

In this case, the possible baskets are classified in Proposition 4.2.14. From the list we can take $m_0 = 8$. And we have $r_X \leq 210$, $-K_X^3 \geq \frac{1}{84}$, and $r_{\max} \leq 14$. By Proposition 4.3.2 with $t = 8$, we can take $m_1 = 38$. Hence by Theorem 4.4.10(ii), φ_{-m} is birational onto its image for all $m \geq 76$.

Case II. $r_{\max} \geq 14$.

Write Reid's basket B_X as

$$\{(b_i, r_i) \mid i = 1, \cdots, s; 0 < b_i \leq \frac{r_i}{2}; b_i \text{ is coprime to } r_i\}.$$

Recall that $r_X = \text{l.c.m.}\{r_i \mid i = 1, \cdots, s\}$ and that

$$\sum_i (r_i - \frac{1}{r_i}) \leq 24$$

by inequality (4.1.1). We recall the sequence $\mathcal{R} = (r_i)_i$ from the proof of Proposition 4.1.1. Denote by $\tilde{r}_1 = r_{\max}$ the largest value in $\mathcal{R}$, by $\tilde{r}_2$ the second largest value, by $\tilde{r}_3$, $\tilde{r}_4$ the third, the forth, and so on. For instance, if $\mathcal{R} = (2, 3, 4, 4, 5, 5)$, then $\tilde{r}_1 = 5$, $\tilde{r}_2 = 4$, $\tilde{r}_3 = 3$, and $\tilde{r}_4 = 2$. If the value $\tilde{r}_j$ does not exist by definition, then we set $\tilde{r}_j = 1$. In the previous example, we have $\tilde{r}_5 = 1$.

Clearly $r_{\max} \leq 24$. We will compute an explicit bound for r_X.

If $r_{\max} \geq 23$, then by inequality (4.1.1), there are no more values in $\mathcal{R}$. Hence $r_X \leq 24$.

If $20 \leq r_{\max} \leq 22$, then by inequality (4.1.1), $\tilde{r}_2 \leq 4$. Hence

$$r_X \leq \text{l.c.m}(r_{\max}, 4, 3, 2) = 132.$$

If $r_{\max} = 19$, then by inequality (4.1.1), $\tilde{r}_2 \leq 5$, and at most one of $3, 4, 5$ can be in $\mathcal{R}$. Hence $r_X \leq 19 \times 5 \times 2 = 190$.

If $r_{\max} = 18$, then by inequality (4.1.1), $\tilde{r}_2 \leq 6$, and at most one of $3, 4, 5, 6$ can be in $\mathcal{R}$. Hence $r_X \leq 18 \times 5 = 90$.

If $r_{\max} = 17$, then by inequality (4.1.1), $\tilde{r}_2 \leq 7$. If $\tilde{r}_2 \geq 5$, then by inequality (4.1.1), $\tilde{r}_3 \leq 2$ and hence $\tilde{r}_X \leq 17 \times 7 \times 2 = 238$. If $\tilde{r}_2 \leq 4$, then $r_X \leq \text{l.c.m}(17, 4, 3, 2) = 204$.

If $r_{\max} = 16$, then by inequality (4.1.1), $\tilde{r}_2 \leq 8$. If $\tilde{r}_2 \geq 6$, then by inequality (4.1.1), $\tilde{r}_3 \leq 2$ and hence $r_X \leq 16 \times 7 = 112$. If $\tilde{r}_2 \leq 5$, then $r_X \leq \text{l.c.m}(16, 5, 4, 3, 2) = 240$.

If $r_{\max} = 15$, then by inequality (4.1.1), $\tilde{r}_2 \leq 9$. If $\tilde{r}_2 \geq 6$, then by inequality (4.1.1), $\tilde{r}_3 \leq 3$ and hence $r_X \leq \text{l.c.m}(r_{\max}, \tilde{r}_2, 3, 2) \leq 15 \times 7 \times 2 = 210$. If $\tilde{r}_2 \leq 5$, then $r_X \leq \text{l.c.m}(15, 5, 4, 3, 2) = 60$.

If $r_{\max} = 14$, then by inequality (4.1.1), $\tilde{r}_2 \leq 10$. If $\tilde{r}_2 \geq 8$, then by inequality (4.1.1), $\tilde{r}_3 \leq 2$ and hence $\tilde{r}_X \leq 14 \times 9 = 126$. If $\tilde{r}_2 \leq 7$, then r_X divides $\text{l.c.m}(14, 6, 5, 4, 3, 2) = 420$. But by inequality (4.1.1), $5, 4, 3$ can not be in $\mathcal{R}$ simultaneously, hence $r_X < 420$. In particular, $r_X \leq 210$.

In summary, when $r_{\max} \geq 14$, we have $r_X \leq 240$.

We can take $m_0 = 8$ by [CC08, Theorem 1.1]. And we have $r_X \leq 240$, $-K_X^3 \geq \frac{1}{240}$ (note that $r_X K_X^3$ is an integer), and $r_{\max} \leq 24$. If $r_{\max} \leq 22$, by Proposition 4.3.2 with $t = 6$, we can take $m_1 = 44$. Hence by Theorem 4.4.10(ii), φ_{-m} is birational onto its image for all $m \geq 96$. If $r_{\max} = 23$ or 24, by Proposition 4.3.2 with $t = 2$, $r_X \leq 24$, $-K_X^3 \geq \frac{1}{24}$, we can take $m_1 = 37$. Hence by Theorem 4.4.10(ii), φ_{-m} is birational onto its image for all $m \geq 93$.

Case III. $r_{\max} < 14$ and $P_{-1} > 0$.

In this case, $\nu_0 = 1$ and by [CC08, Theorem 1.1], we can take $m_0 = 8$.

If $r_X \leq 660$ and $r_{\max} \leq 12$, then by Proposition 4.3.2 with $t = 15$, $r_{\max} \leq 12$, and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 65$. Hence by Theorem 4.4.10(iii), φ_{-m} is birational onto its image for all $m \geq 97$.

If $r_X \leq 660$ and $r_{\max} = 13$, Then $\tilde{r}_2 \leq 11$. If $\tilde{r}_2 \geq 9$, then $\tilde{r}_3 \leq 2$ and $r_X \leq 286$. If $\tilde{r}_2 = 8$, then $\tilde{r}_3 \leq 3$ and $r_X \leq 312$. If $\tilde{r}_2 = 7$, then $\tilde{r}_3 \leq 4$ and $3, 4$ can not be in $\mathcal{R}$ simultaneously, hence $r_X \leq 546$. If $\tilde{r}_2 \leq 6$, then r_X divides 780 and hence $r_X \leq 390$ by Proposition 4.1.1. In summary, $r_X \leq 546$. By Proposition 4.3.2 with $t = 10$, $r_{\max} = 13$, and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 61$. Hence by Theorem 4.4.10(iii), φ_{-m} is birational onto its image for all $m \geq 95$.

If $r_X > 660$, then $r_X = 840$ and $r_{\max} = 8$. By Theorem 1.2.14, we can take $m_1 = 71$. Hence by Theorem 4.4.10(iii), φ_{-m} is birational onto its image for all $m \geq 95$.

Case IV. $r_{\max} < 14$, $P_{-1} = 0$, and $P_{-2} > 0$.

In this case, $\nu_0 = 2$ and by [CC08, Proposition 3.10, Case 1], we can take

$m_0 = 6$.

If $P_{-4} = 1$, then $P_{-2} = 1$. By the proof of Theorem 4.2.12 (note that the arguments on baskets are valid without assuming $\rho = 1$), we are exactly in the situation $(P_{-3}, P_{-4}) = (0, 1)$, corresponding to the last paragraph of Subsubcase II-3-iii of Theorem 4.2.12. In fact, the possible baskets are classified in the following list:

$$\begin{aligned}
&\{9 \times (1,2), (1,3), (1,7)\},\\
&\{8 \times (1,2), (2,5), (1,7)\},\\
&\{8 \times (1,2), (2,5), (1,6)\},\\
&\{7 \times (1,2), (3,7), (1,6)\},\\
&\{6 \times (1,2), (4,9), (1,6)\},\\
&\{7 \times (1,2), (3,7), (1,5)\},\\
&\{6 \times (1,2), (4,9), (1,5)\},\\
&\{5 \times (1,2), (5,11), (1,5)\},\\
&\{4 \times (1,2), (6,13), (1,5)\}.
\end{aligned}$$

Hence in this case $r_X \leq 130$, $-K_X^3 \geq \frac{1}{130}$, and $r_{\max} \leq 13$. By Proposition 4.3.2 with $t = 7$, we can take $m_1 = 37$. Hence by Theorem 4.4.10(iii), φ_{-m} is birational onto its image for all $m \geq 95$.

Hence, from now on, we assume that $P_{-4} > 1$. So we may take $m_0 = 4$.

If $r_{\max} \leq 8$, then r_X divides $\mathrm{l.c.m}(8, 7, 6, 5, 4, 3, 2) = 840$. Suppose $r_X < 840$, then $r_X \leq 420$. By Proposition 4.3.2 with $t = 20$ and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 54$. Hence by Theorem 4.4.10(iii), φ_{-m} is birational onto its image for all $m \geq 90$. Suppose $r_X = 840$, then $\mathcal{R} = (3, 5, 7, 8)$ or $(2, 3, 5, 7, 8)$ as we have seen in the proof of Proposition 4.1.1. However,

$$\begin{aligned}
P_{-1} &= \frac{1}{2}(-K_X^3) - \sum \frac{b_i(r_i - b_i)}{2r_i} + 3\\
&> 3 - \frac{1}{4} - \frac{2}{6} - \frac{6}{10} - \frac{12}{14} - \frac{15}{16} > 0, \qquad (4.4.9)
\end{aligned}$$

a contradiction.

The above argument reminds us to find a condition corresponding to $P_{-1} = 0$. Assume that 2 is not in $\mathcal{R}$, then

$$\begin{aligned}
P_{-1} &= \frac{1}{2}(-K_X^3) - \sum \frac{b_i(r_i - b_i)}{2r_i} + 3\\
&> 3 - \frac{1}{8} \sum (r_i - \frac{1}{r_i}) \geq 0,
\end{aligned}$$

a contradiction. Hence, $2 \in \mathcal{R}$.

Consider the case $r_{\max} = 9$. If $\tilde{r}_2 \leq 6$, then $r_X \leq \mathrm{l.c.m}(9,6,5,4,3,2) = 180$. If $\tilde{r}_2 = 8$, then by inequality (4.1.1) and $2 \in \mathcal{R}$, $\tilde{r}_2 \leq 5$ and $r_X \leq \mathrm{l.c.m}(9,8,5,4,3,2) = 360$. If $\tilde{r}_2 = 7$ and $5 \notin \mathcal{R}$, then

$$r_X \leq \mathrm{l.c.m.}(9,7,6,4,3,2) = 252.$$

If $\tilde{r}_2 = 7$ and $5 \in \mathcal{R}$, then $6 \notin \mathcal{R}$ and r_X divides $\mathrm{l.c.m}(9,7,5,4,3,2) = 630$. In summary, $r_X \leq 360$ or $r_X = 630$. Whenever $r_X \leq 360$, by Proposition 4.3.2 with $t = 12$ and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 50$. Hence by Theorem 4.4.10(iii), φ_{-m} is birational onto its image for all $m \geq 90$. Whenever $r_X = 630$, then $2, 5, 7, 9$ must be in $\mathcal{R}$. Hence $\mathcal{R} = (2,5,7,9)$ or $(2,2,5,7,9)$ by inequality (4.1.1). In this case, arguing as inequality (4.4.9), B_X can only be $\{2 \times (1,2), (2,5), (3,7), (4,9)\}$. We will choose suitable m_1 and modify the upper bound of μ_0. Since $P_{-4} = 2$, $|-4K_X|$ is composed with a pencil. Note that $P_{-7} = 10$ and $P_{-3} = 1$. If $|-7K_X|$ is not composed with a pencil, then we can take $m_1 = 7$. By Theorem 4.4.10(ii), φ_{-m} is birational onto its image for all $m \geq 29$. If $|-7K_X|$ is also composed with a pencil, then we know $\mu_0 \leq \frac{7}{9}$ by Remark 4.4.2. Also we can see $P_{-61} = 5294 > r_X(-K_X^3)61 + 1$ by direct computation where $-K_X^3 = \frac{43}{315}$. Hence we can take $m_1 = 61$ by Corollary 4.3.1. Hence by Theorem 4.4.10(iii), φ_m is birational for all $m \geq 97$.

Consider the case $r_{\max} = 10$. If $\tilde{r}_2 \leq 6$, then $r_X \leq \mathrm{l.c.m}(10,6,5,4,3,2) = 60$. If $\tilde{r}_2 = 7$, then r_X divides $\mathrm{l.c.m}(10,7,5,4,3,2) = 420$, but $3, 4$ can not be in $\mathcal{R}$ simultaneously, hence $r_X \leq 210$. If $\tilde{r}_2 = 8$, then $r_3 \leq 4$ and $r_X \leq \mathrm{l.c.m}(10,8,4,3,2) = 120$. If $\tilde{r}_2 = 9$, then $\tilde{r}_3 \leq 3$ and $r_X \leq \mathrm{l.c.m}(10,9,3,2) = 90$. Hence in summary, $r_X \leq 210$. By Proposition 4.3.2 with $t = 10$ and $-K_X^3 \geq \frac{1}{210}$, we can take $m_1 = 39$. Hence by Theorem 4.4.10(ii), φ_{-m} is birational onto its image for all $m \geq 71$.

Consider the case $r_{\max} = 11$. If $\tilde{r}_2 = 10$, then $\tilde{r}_3 \leq 2$ and $r_X \leq 110$. If $\tilde{r}_2 = 9$ or 8, then $\tilde{r}_3 \leq 3$ and $r_X \leq 264$. If $\tilde{r}_2 = 7$, then $\tilde{r}_3 \leq 4$ and $3, 4$ can not be in $\mathcal{R}$ simultaneously, hence $r_X \leq 308$ or $r_X = \mathrm{l.c.m}(11,7,3,2) = 462$. If $\tilde{r}_2 = 6$, then $5, 4$ can not be in $\mathcal{R}$ simultaneously, hence $r_X \leq \mathrm{l.c.m}(11,6,5,3,2) = 330$. If $\tilde{r}_2 \leq 5$, then r_X divides $\mathrm{l.c.m}(11,5,4,3,2) = 660$. In summary, $r_X \leq 330$ or $r_X = 462$ or $r_X = 660$. Whenever $r_X = 660$, then $2, 3, 4, 5, 11$ must be in $\mathcal{R}$. Hence $\mathcal{R} = (2,3,4,5,11)$ by inequality (4.1.1). Arguing as inequality (4.4.9), this implies $P_{-1} > 0$, a contradiction. Whenever $r_X \leq 330$, by Proposition 4.3.2 with $t = 13$ and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 48$. Hence by Theorem 4.4.10(ii), φ_{-m} is birational onto its image for all $m \geq 86$. If $r_X = 462$, then $2, 3, 7, 11$ must be in $\mathcal{R}$. Hence $\mathcal{R} = (2,3,7,11)$ or $(2,2,3,7,11)$ by inequality (4.1.1). Arguing as inequality (4.4.9), B_X can only be $\{2 \times (1,2),(1,3),(3,7),(5,11)\}$. In this case we can prove $P_{-52} = 2612 > r_X(-K_X^3)52+1$ by direct computation where $-K_X^3 = \frac{50}{462}$. Hence we can take $m_1 = 52$. By Theorem 4.4.10(ii), φ_{-m} is birational onto its image for all $m \geq 93$.

Consider the case $r_{\max} = 12$. Then $\tilde{r}_2 \leq 10$ and at most one of $5, 6, 7, 8, 9, 10$ will be in $\mathcal{R}$. Hence $r_X \leq 84$. By Proposition 4.3.2 with $t = 5$ and $-K_X^3 \geq \frac{1}{84}$, we can take $m_1 = 37$. Hence by Theorem 4.4.10(ii), φ_{-m} is birational onto its image for all $m \geq 68$.

Finally, consider the case $r_{\max} = 13$. Then $\tilde{r}_2 \leq 9$. If $\tilde{r}_2 = 9$ or 8, then $\tilde{r}_3 \leq 2$ and $r_X \leq 234$. If $\tilde{r}_2 = 7$, then $\tilde{r}_3 \leq 3$ and $r_X = 546$ or 182. If $\tilde{r}_2 \leq 6$, then r_X divides 780 and hence $r_X \leq 390$ by Proposition 4.1.1. In summary, $r_X \leq 390$ or $r_X = 546$. Whenever $r_X \leq 390$, by Proposition 4.3.2 with $t = 12$ and $-K_X^3 \geq \frac{1}{330}$, we can take $m_1 = 52$. Hence by Theorem 4.4.10(ii), φ_{-m} is birational onto its image for all $m \geq 93$. Whenever $r_X = 546$, then $\mathcal{R} = (2, 3, 7, 13)$. Argue as inequality (4.4.9), B_X can only be $\{(1,2),(1,3),(3,7),(6,13)\}$. We will choose suitable m_1 and modify the upper bound of μ_0. Since $P_{-4} = 2$, $|-4K_X|$ is composed with a pencil. Note that $P_{-10} = 21$ and $P_{-6} = 5$. If $|-10K_X|$ is not composed with a pencil, then we can take $m_1 = 10$. By Theorem 4.4.10(ii), φ_{-m} is birational onto its image for all $m \geq 40$. If $|-10K_X|$ is also composed with a pencil, then we know $\mu_0 \leq \frac{1}{2}$ by Remark 4.4.2. Also we can prove $P_{-57} = 3540 > r_X(-K_X^3)57+1$ by direct computation where $-K_X^3 = \frac{61}{546}$. Hence we can take $m_1 = 57$. By Theorem 4.4.10(ii), φ_{-m} is birational onto its image for all $m \geq 95$. $\square$

5

On birational geometry of minimal threefolds with numerically trivial canonical divisors

In this chapter, we investigate minimal 3-folds with $K \equiv 0$. We will prove Theorems 1.2.17 and 1.2.18.

For the convenience, we introduce the following definition.

Definition 5.0.14. (X, L, T) is called a *polarized triple* if X is a minimal 3-fold with $q(X) = 0$ and $K_X \equiv 0$, L is a nef and big Weil divisor, and T is a numerically trivial Weil divisor on X.

Note that we assume $q(X) = 0$ in the definition. The case that $q(X) > 0$ is relatively easy and we treat it in Section 5.2 (see Theorem 5.2.2).

This chapter is organized as follows. We collect some facts in Section 5.1. We treat the Gorenstein case as a generalization of Fukuda and Oguiso–Peternell's results in Section 5.2. We study the birationality of polarized triples in Section 5.3 and give an effective criterion for the birationality of linear systems. In the last section, to apply the birationality criterion, we estimate several quantities of polarized triples. As applications, we prove Theorems 1.2.17 and 1.2.18.

5.1 Some facts about minimal 3-folds with $K \equiv 0$

We collect some facts about minimal 3-folds with $K \equiv 0$ proved by Kawamata [Kaw86] and Morrison [Mor86].

By Kawamata [Kaw85, Theorem 8.2], $K_X \sim_{\mathbb{Q}} 0$ and we define the *global index*

$$I(X) = \min\{m \in \mathbb{N} \mid mK_X \sim 0\}.$$

Note that $i(X)|I(X)$.

Theorem 5.1.1 ([Kaw86, Mor86])**.** *Let X be a minimal 3-fold with $K_X \equiv 0$. The following facts hold:*

(i) $0 \leq \chi(\mathcal{O}_X) \leq 4$*;*

(ii) $\chi(\mathcal{O}_X) = 0$ *if and only if X has Gorenstein singularities;*

(iii) If $q(X) > 0$, then X is smooth;

(iv) If $q(X) = 0$ and $\chi(\mathcal{O}_X) \geq 2$, then $I(X) \in \{2, 3, 4, 6\}$;

(v) If $q(X) = 0$ and $\chi(\mathcal{O}_X) = 1$, then

$$I(X) \in \{2, 3, 4, 5, 6, 8, 10, 12\};$$

(vi) If $I(X) \in \{5, 8, 10, 12\}$, then $\chi(\mathcal{O}_X) = 1$, $q(X) = h^2(\mathcal{O}_X) = 0$, $i(X) = I(X)$, and the singular points can be described explicitly by Morrison [Mor86, Proposition 3].

Proof. (i) is proved by Kawamata [Kaw86, Theorem 3.1]. (ii) is a direct consequence of equality (2.5.1). (iii) is proved by Kawamata [Kaw85, Kaw86] (see [Mor86, Section 1]). (iv) is proved by Morrison [Mor86, Proposition 1, Proof of Theorem 1] and (v) is proved by Morrison [Mor86, Proposition 3, Proof of Theorem 2]. (vi) is a direct consequence of (ii)-(v) and Morrison [Mor86, Proposition 3]. □

5.2 Gorenstein case

Throughout this section, we assume that X is a minimal Gorenstein 3-fold with $K_X \equiv 0$ and L is a nef and big Weil divisor on X. Note that L is a Cartier divisor since $i(X) = 1$. Recall that we have a *canonical model* $\mu : (X, L) \to (Z, H)$ such that Z is a 3-fold with canonical singularities and $\mu^* K_Z = K_X$, H is an ample Catier divisor with $L = \mu^* H$ (cf. [OP95, Lemma 0.2]).

Lemma 5.2.1 (cf. [OP95, Lemma 1.1])**.** *Let D be a divisor on X. Then*

(i) $(D \cdot L^2)^2 \geq (D^2 \cdot L)(L^3)$*;*

(ii) $D \cdot L^2 \equiv D^2 \cdot L \mod 2$;

(iii) If $D \cdot L^2 = 1$ *and* $D^2 \cdot L \geq 0$, *then* $D^2 \cdot L = L^3 = 1$.

Proof. See the proof of [OP95, Lemma 1.1]. Note that $K_X \equiv 0$ is sufficient in the proof. □

We prove Theorem 1.2.17 for the Gorenstein case. It is a direct generalization of Fukuda [Fuk91] and Oguiso–Peternell [OP95], and we follow their ideas.

Theorem 5.2.2. *Let X be a minimal Gorenstein 3-fold with $K_X \equiv 0$, a nef and big Weil divisor L, and a Weil divisor $T \equiv 0$. Then $|K_X + mL + T|$ gives a birational map for all $m \geq 5$.*

Proof. Note that L and T are Cartier divisors since $i(X) = 1$.

Case 1. $\dim \Phi_{|L|}(X) \geq 1$.

Take a resolution $\pi : Y \to X$. Consider the linear system $|K_Y + m\pi^*L + \pi^*T|$. Note that

$$\dim \Phi_{|\pi^*L|}(Y) = \dim \Phi_{|L|}(X) \geq 1.$$

By [Fuk91, Key Lemma] with $R = \pi^*L$, $r_0 = 4$, and $r_1 = 1$, $|K_Y + m\pi^*L + \pi^*T|$ gives a birational map for all $m \geq 5$. So $|K_X + mL + T|$ gives a birational map for all $m \geq 5$.

Case 2. $\dim \Phi_{|L|}(X) \leq 0$.

In this case, since $h^0(L) > 0$ by Riemann–Roch formula, we have $h^0(L) = 1$. By Riemann–Roch formula again,

$$h^0(2L) = \frac{1}{6}(2^3 - 2)L^3 + 2h^0(L) = L^3 + 2.$$

First, we assume that $|2L|$ is composed with a pencil of surfaces. Set $D := 2L$ and keep the same notation as in Section 2.4. Then we have

$$2\pi^*(L) \geq M \equiv aS \geq (h^0(2L) - 1)S = (L^3 + 1)S.$$

Thus we have $2L^3 \geq (L^3 + 1)(\pi^*(L)^2 \cdot S)$. This implies that $L^2 \cdot \pi_*S = \pi^*(L)^2 \cdot S = 1$ since $\pi^*(L)^2 \cdot S > 0$. On the other hand, $L \cdot (\pi_*S)^2 = \pi^*(L) \cdot \pi^*\pi_*S \cdot S \geq 0$. Hence by Lemma 5.2.1(iii), $L \cdot (\pi_*S)^2 = L^3 = 1$. And hence $M \equiv (L^3 + 1)S = 2S$, in particular, $|2L|$ is composed with a rational pencil. Consider the canonical model (Z, H). Since $h^0(H) = h^0(L) = 1$ and $H^3 = L^3 = 1$, there exists an irreducible surface G such that $|H| = \{G\}$. Denote by G' the strict transform G. Then we may write $2L \sim 2G' + 2E$ for some μ-exceptional divisor E. Note that $\mathrm{Mov}|2L| = |2\pi_*S|$, hence $\pi_*S \sim$

$G' + E'$ for some μ-exceptional divisor E'. But this implies $\dim|\pi_*S| = 0$, a contradiction.

Hence $|2L|$ is not composed with a pencil of surface. Set $D := 2L$ and keep the same notation as in Section 2.4. Then we have $2\pi^*(L) = |M| + F$ such that $|M|$ is base point free. Consider a smooth element N in $|M|$. Note that $|K_X + mL + T|$ gives a birational map if so does the restriction $|K_Y + N + \pi^*((m-2)L + T)||_N$ by Lemma 2.7.2 and birationality principle (cf. [OP95, Lemma 1.3]). On the other hand, Kawamata–Viehweg vanishing theorem and adjunction formula give

$$|K_Y + N + \pi^*((m-2)L + T)||_N = |K_N + \pi^*((m-2)L + T)|_N|.$$

Reider's theorem (cf. [Reider88]) implies that $|K_N + \pi^*((m-2)L + T)|_N|$ gives a birational map for $m \geq 5$ if $(\pi^*L)^2 \cdot N \geq 2$. Now we assume that $(\pi^*L)^2 \cdot N = 1$, then Lemma 5.2.1(iii) implies that $L^3 = L^2 \cdot \pi_*N = L \cdot (\pi_*N)^2 = 1$. Consider the canonical model (Z, H). Since $h^0(H) = H^3 = 1$, and $L^2 \cdot \pi_*N = 1$, a similar argument implies $\dim|\pi_*N| = 0$, a contradiction.

We completed the proof. $\square$

By Theorems 5.2.2 and 5.1.1(ii)(iii), to prove Theorem 1.2.17, we only need to consider polarized triples (X, L, T) with $\chi(\mathcal{O}_X) > 0$.

5.3 Birationality criterion

In this section, we give a criterion for the birationality of polarized triples.

5.3.1 Key theorem

Let (X, L, T) be a polarized triple. Take a Weil divisor L_0 such that $L_0 \equiv L$. Suppose that $h^0(m_0L_0) \geq 2$ for some integer $m_0 > 0$. Suppose that $m_1 \geq m_0$ is an integer with $h^0(m_1L_0) \geq 2$ and that $|m_1L_0|$ and $|m_0L_0|$ are not composed with the same pencil.

Set $D := m_0L_0$ and keep the same notation as in Section 2.4. We may modify the resolution π in Section 2.4 such that the movable part $|M_m|$ of $|\lfloor\pi^*(mL_0)\rfloor|$ is base point free for all $m_0 \leq m \leq m_1$. Set $\iota_m := \iota(mL_0)$ defined in Section 2.4. Recall that, for any integer m with $h^0(mL_0) > 1$,

$$\iota_m = \begin{cases} 1, & \text{if } |mL_0| \text{ is not composed with a pencil;} \\ h^0(mL_0) - 1, & \text{if } |mL_0| \text{ is composed with a pencil.} \end{cases}$$

Pick a generic irreducible element S of $|M_{m_0}|$. We have

$$m_0\pi^*(L_0) = \iota_{m_0}S + F_{m_0}$$

for some effective $\mathbb{Q}$-divisor F_{m_0}. In particular, we see that

$$\frac{m_0}{\iota_{m_0}}\pi^*(L_0) - S \sim_{\mathbb{Q}} \text{effective } \mathbb{Q}\text{-divisor}.$$

Define the real number

$$\mu_0 = \mu_0(|S|) := \inf\{t \in \mathbb{Q}^+ \mid t\pi^*(L_0) - S \sim_{\mathbb{Q}} \text{effective } \mathbb{Q}\text{-divisor}\}.$$

Remark 5.3.1. Clearly, we have $0 < \mu_0 \leq \frac{m_0}{\iota_{m_0}} \leq m_0$. For all k such that $|kL_0|$ and $|m_0L_0|$ are composed with the same pencil, we have

$$k\pi^*(L_0) = \iota_k S + F_k$$

for some effective $\mathbb{Q}$-divisor F_k, and hence $\mu_0 \leq \frac{k}{\iota_k}$.

By our assumption on $|m_1L_0|$, we know that $|G| = |M_{m_1}|_S|$ is a base point free linear system on S and $h^0(S, G) \geq 2$. Denote by C a generic irreducible element of $|G|$. Note that since $K_X \equiv 0$, K_Y is pseudo-effective and hence $g(C) \geq 1$. Since $m_1\pi^*(L_0) \geq M_{m_1}$, we have

$$m_1\pi^*(L_0)|_S \equiv C + H$$

where H is an effective $\mathbb{Q}$-divisor on S.

We define two numbers which will be the key invariants accounting for the birationality of $\Phi_{|K_X+mL+T|}$. They are

$$\begin{aligned} \zeta &:= (\pi^*(L)\cdot C)_Y = (\pi^*(L_0)\cdot C)_Y = (\pi^*(L_0)|_S \cdot C)_S \text{ and} \\ \epsilon(m) &:= (m - \mu_0 - m_1)\zeta. \end{aligned}$$

Note that ζ and $\epsilon(m)$ are birational invariants by projection formula. Hence we can modify π if necessary. Also note that $\zeta > 0$ since L is nef and big and C is free.

While studying the birationality of $\Phi_{|K_X+mL+T|}$, we always require that the linear system $\Lambda_m := |K_Y + \lceil \pi^*(mL+T)\rceil|$ satisfies the following assumption for some integer $m > 0$.

Assumption 5.3.2. Keep the notation as above.

(1) The linear system Λ_m distinguishes different generic irreducible elements of $|M_{m_0}|$ (namely, $\Phi_{\Lambda_m}(S') \neq \Phi_{\Lambda_m}(S'')$ for two different generic irreducible elements S', S'' of $|M_{m_0}|$).

(2) The linear system $\Lambda_m|_S$ distinguishes different generic irreducible elements of the linear system $|G| = |M_{m_1}|_S|$ on S.

The following is our key theorem.

Theorem 5.3.3. *Let (X, L, T) be a polarized triple. Keep the notation as above. Let $m > 0$ be an integer. If Assumption 5.3.2 is satisfied and $\epsilon(m) > 2$, then $\Phi_{|K_X+mL+T|}$ is birational onto its image.*

Proof. By Lemma 2.7.2, we only need to prove the birationality of Φ_{Λ_m}. Since Assumption 5.3.2(1) is satisfied, the usual birationality principle reduces the birationality of Φ_{Λ_m} to that of $\Phi_{\Lambda_m}|_S$ for a generic irreducible element S of $|M_{m_0}|$. Similarly, due to Assumption 5.3.2(2), we only need to prove the birationality of $\Phi_{\Lambda_m}|_C$ for a generic irreducible element C of $|G|$. Now we show how to restrict the linear system Λ_m to C.

Now assume $\epsilon(m) > 0$. We can find a sufficiently large integer n so that there exists a number $\mu_n \in \mathbb{Q}^+$ with $0 \leq \mu_n - \mu_0 \leq \frac{1}{n}$, $\lceil \epsilon(m, n) \rceil = \lceil \epsilon(m) \rceil$ where $\epsilon(m, n) := (m - \mu_n - m_1)\zeta$, and

$$\mu_n \pi^*(L_0) \sim_{\mathbb{Q}} S + E_n$$

for an effective $\mathbb{Q}$-divisor E_n. In particular, $\epsilon(m, n) > 0$, and $\epsilon(m, n) > 2$ if $\epsilon(m) > 2$. Re-modify our π in Section 2.4 so that E_n has simple normal crossing support.

For the given integer $m > 0$, we have

$$|K_Y + \lceil \pi^*(mL + T) - E_n \rceil| \preceq |K_Y + \lceil \pi^*(mL + T) \rceil|. \tag{5.3.1}$$

Since $\epsilon(m, n) > 0$, the $\mathbb{Q}$-divisor

$$\pi^*(mL + T) - E_n - S \equiv (m - \mu_n)\pi^*(L)$$

is nef and big and thus

$$H^1(Y, K_Y + \lceil \pi^*(mL + T) - E_n \rceil - S) = 0$$

by Kawamata–Viehweg vanishing theorem. Hence we have surjective map

$$H^0(Y, K_Y + \lceil \pi^*(mL + T) - E_n \rceil) \longrightarrow H^0(S, K_S + L_{m,n}) \tag{5.3.2}$$

where

$$L_{m,n} := (\lceil \pi^*(mL + T) - E_n \rceil - S)|_S \geq \lceil \mathcal{L}_{m,n} \rceil \tag{5.3.3}$$

and $\mathcal{L}_{m,n} := (\pi^*(mL + T) - E_n - S)|_S$. Moreover, we have

$$m_1 \pi^*(L_0)|_S \equiv C + H$$

for an effective $\mathbb{Q}$-divisor H on S by the setting. Thus the $\mathbb{Q}$-divisor

$$\mathcal{L}_{m,n} - H - C \equiv (m - \mu_n - m_1)\pi^*(L)|_S$$

is nef and big by $\epsilon(m,n) > 0$. And by Kawamata–Viehweg vanishing theorem again,

$$H^1(S, K_S + \lceil \mathcal{L}_{m,n} - H \rceil - C) = 0.$$

Therefore, we have surjective map

$$H^0(S, K_S + \lceil \mathcal{L}_{m,n} - H \rceil) \longrightarrow H^0(C, K_C + D_{m,n}) \tag{5.3.4}$$

where

$$D_{m,n} := \lceil \mathcal{L}_{m,n} - H - C \rceil|_C \geq \lceil \mathcal{D}_{m,n} \rceil \tag{5.3.5}$$

and $\mathcal{D}_{m,n} := (\mathcal{L}_{m,n} - H - C)|_C$ with $\deg \mathcal{D}_{m,n} = \epsilon(m,n)$.

Now by inequalities (5.3.1), (5.3.3), (5.3.5), and surjective maps (5.3.2), (5.3.4), to prove the birationality of $\Phi_{\Lambda_m}|_C$, it is sufficient to prove that $|K_C + \lceil \mathcal{D}_{m,n} \rceil|$ gives a birational map. Clearly this is the case whenever $\epsilon(m) > 2$, which in fact implies $\deg(\lceil \mathcal{D}_{m,n} \rceil) \geq \lceil \epsilon(m,n) \rceil \geq 3$ and $K_C + \lceil \mathcal{D}_{m,n} \rceil$ is very ample. We complete the proof. □

Corollary 5.3.4. *Keep the same notation as above. For any integer $m > 0$, set*

$$\epsilon(m,0) := (m - \frac{m_0}{\iota_{m_0}} - m_1)\zeta.$$

If $\epsilon(m,0) > 0$, then

$$\Lambda_m|_S \succeq |K_S + L_m|$$

where $L_m := (\lceil \pi^*(mL+T) - \frac{1}{\iota_{m_0}} F_{m_0} \rceil - S)|_S$.

Proof. Recall that

$$m_0 \pi^*(L_0) = \iota_{m_0} S + F_{m_0}.$$

First of all, relation (5.3.1) reads

$$|K_Y + \lceil \pi^*(mL+T) - \frac{1}{\iota_{m_0}} F_{m_0} \rceil| \preceq |K_Y + \lceil \pi^*(mL+T) \rceil|.$$

In fact, as long as $\epsilon(m,0) > 0$, the front part of the proof of Theorem 5.3.3 is valid. In explicit, subjective map (5.3.2) reads the following surjective map

$$H^0(Y, K_Y + \lceil \pi^*(mL+T) - \frac{1}{\iota_{m_0}} F_{m_0} \rceil) \longrightarrow H^0(S, K_S + L_m)$$

where

$$L_m = (\lceil \pi^*(mL+T) - \frac{1}{\iota_{m_0}} F_{m_0} \rceil - S)|_S.$$

Hence we have proved the statement. □

5.3.2 Criterion

In order to apply Theorem 5.3.3, we need to verify Assumption 5.3.2 and $\epsilon(m) > 2$ in advance, for which one of the crucial steps is to estimate the lower bound of ζ.

Proposition 5.3.5. *Let $m > 0$ be an integer. Keep the same notation as in Subsection 5.3.1. Then*

(i) If $\epsilon(m) > 1$, then $\zeta \geq \frac{2g(C)-2+\lceil \epsilon(m) \rceil}{m}$;

(ii) Moreover,

$$\zeta \geq \frac{2g(C)-1}{\mu_0 + m_1 + 1};$$

(iii) If $g(C) = 1$, then $\zeta \geq 1$;

(iv) $i(X)\zeta \in \mathbb{Z}_{>0}$.

In summary,

$$\zeta \geq \left\lceil i(X) \min\left\{1, \frac{3}{\mu_0 + m_1 + 1}\right\}\right\rceil / i(X).$$

Proof. (i). Recall that since $K_X \equiv 0$, K_Y is pseudo-effective and hence $g(C) \geq 1$. In the proof of Theorem 5.3.3, if $\epsilon(m) > 1$ then $|K_C + \lceil \mathcal{D}_{m,n} \rceil|$ is base point free with

$$\deg(K_C + \lceil \mathcal{D}_{m,n} \rceil) \geq 2g(C) - 2 + \lceil \epsilon(m,n) \rceil = 2g(C) - 2 + \lceil \epsilon(m) \rceil.$$

Denote by $\mathcal{N}_m$ the movable part of $|K_S + \lceil \mathcal{L}_{m,n} - H \rceil|$. Note that

$$\begin{aligned} &H^0(\mathcal{O}_X(K_X + mL + T)) \\ \cong\ &H^0(\mathcal{O}_Y(\lfloor \pi^*(K_X + mL + T) \rfloor)) \\ \cong\ &H^0(\mathcal{O}_Y(K_Y + \lceil \pi^*(mL + T) \rceil)). \end{aligned}$$

Denote by $\mathcal{M}_m$ the movable part of $|\lfloor \pi^*(K_X + mL + T) \rfloor|$. Noting the relations (5.3.1), (5.3.2), and $|K_S + \lceil \mathcal{L}_{m,n} - H \rceil| \preceq |K_S + \lceil \mathcal{L}_{m,n} \rceil|$ while applying [Chen01, Lemma 2.7], we get

$$\pi^*(K_X + mL + T)|_S \geq \mathcal{M}_m|_S \geq \mathcal{N}_m$$

and $\mathcal{N}_m|_C \geq K_C + \lceil \mathcal{D}_{m,n} \rceil$ since the latter one is base point free. So we have

$$m\zeta = \pi^*(K_X + mL + T)|_S \cdot C \geq \mathcal{N}_m \cdot C \geq \deg(K_C + \lceil \mathcal{D}_{m,n} \rceil).$$

Hence

$$m\zeta \geq 2g(C) - 2 + \lceil \epsilon(m) \rceil.$$

(ii). Take $m' = \min\{m \mid \epsilon(m) > 1\}$, then (i) implies $\zeta \geq \frac{2g(C)}{m'}$. We may assume that $m' > \mu_0 + m_1 + 1$ otherwise $\zeta \geq \frac{2g(C)}{\mu_0+m_1+1}$. Hence

$$\begin{aligned}\epsilon(m'-1) &= (m' - 1 - \mu_0 - m_1)\zeta \\ &\geq (m' - 1 - \mu_0 - m_1)\frac{2g(C)}{m'}.\end{aligned}$$

By the minimality of m', it follows that $\epsilon(m'-1) \leq 1$. Hence $m' \leq \frac{2g(C)}{2g(C)-1}(\mu_0 + m_1 + 1)$. Then

$$\zeta \geq \frac{2g(C)}{m'} \geq \frac{2g(C) - 1}{\mu_0 + m_1 + 1}.$$

(iii). Recall that

$$K_Y = \pi^* K_X + E_\pi \equiv E_\pi,$$

where E_π is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor whose support contains all π-exceptional divisors since X has at worst terminal singularities. If $g(C) = 1$, then

$$\begin{aligned}0 &= ((K_S + C) \cdot C)_S \\ &= (K_Y \cdot C)_Y + (S \cdot C)_Y + (C^2)_S \\ &= (E_\pi \cdot C)_Y + (S \cdot C)_Y + (C^2)_S.\end{aligned}$$

Since C is free on a free surface S, $(C^2)_S$, $(S \cdot C)_Y$, and $(E_\pi \cdot C)_Y$ are non-negative. Hence $(E_\pi \cdot C)_Y = 0$, which implies that $(E \cdot C)_Y = 0$ for any π-exceptional divisor E on Y since X has at worst terminal singularities. Hence $\zeta = (\pi^* L \cdot C)_Y$ is an integer. On the other hand, $\zeta > 0$. Hence $\zeta \geq 1$.

(iv). It follows from the fact that $i(X)L$ is Cartier.

In summary, if $g(C) = 1$, by (iii), $\zeta \geq 1$; if $g(C) \geq 2$, by (ii), $\zeta \geq \frac{3}{\mu_0+m_1+1}$. Then by (iv),

$$i(X)\zeta \geq \left\lceil i(X) \min\left\{1, \frac{3}{\mu_0 + m_1 + 1}\right\}\right\rceil.$$

We complete the proof. □

Define

$$\rho_0 := \min\{k \in \mathbb{Z}_{>0} \mid h^0(mL + T') > 0 \text{ for all } m \geq k \text{ and for all } T' \equiv 0\}.$$

To verify Assumption 5.3.2, we have the following propositions.

Proposition 5.3.6. *Let (X, L, T) be a polarized triple. Keep the same notation as Subsection 5.3.1. Then Assumption 5.3.2(1) is satisfied for all $m \geq m_0 + \rho_0$.*

Proof. We have

$$\begin{aligned}
&K_Y + \lceil \pi^*(mL + T) \rceil \\
&\geq K_Y + \lceil \pi^*(mL + T - m_0 L_0) + M_{m_0} \rceil \\
&= (K_Y + \lceil \pi^*(mL + T - m_0 L_0) \rceil) + M_{m_0} \\
&\geq M_{m_0}.
\end{aligned}$$

The last inequality is due to

$$\begin{aligned}
&h^0(K_Y + \lceil \pi^*(mL + T - m_0 L_0) \rceil) \\
&= h^0(K_X + mL + T - m_0 L_0)) > 0
\end{aligned}$$

when $m \geq m_0 + \rho_0$ by the definition of ρ_0.

When $f : Y \to \Gamma$ is of type (f_{np}), [Tan71, Lemma 2] implies that Λ_m can distinguish different generic irreducible elements of $|M_{m_0}|$. When f is of type (f_{p}), since the rational pencil $|M_{m_0}|$ (recall that $q(X) = 0$) can already separate different fibers of f, Λ_m can naturally distinguish different generic irreducible elements of $|M_{m_0}|$. □

Proposition 5.3.7. *Let (X, L, T) be a polarized triple. Keep the same notation as in Subsection 5.3.1. Then Assumption 5.3.2(2) is satisfied for all $m \geq m_0 + m_1 + \rho_0$.*

Proof. Assuming $m \geq m_0 + m_1 + 1$, we have $\epsilon(m, 0) > 0$, and Corollary 5.3.4 implies that

$$\Lambda_m|_S \succeq |K_S + L_m|.$$

It suffices to prove that $|K_S + L_m|$ can distinguish different generic irreducible elements of $|G|$.

For a suitable integer $m > 0$, we have

$$\begin{aligned}
&K_S + L_m \\
&= K_Y|_S + \lceil \pi^*(mL + T) - \frac{1}{\iota_{m_0}} F_{m_0} \rceil|_S \\
&\geq (K_Y + \lceil \pi^*(mL + T - (m_0 + m_1)L_0) \rceil)|_S + M_{m_1}|_S.
\end{aligned}$$

Thus, if $|G|$ is not composed with an irrational pencil of curves, $|K_S + L_m|$ can distinguish different irreducible elements provided that

$$K_Y + \lceil \pi^*(mL + T - (m_0 + m_1)L_0) \rceil$$

is effective, which holds for $m - m_0 - m_1 \geq \rho_0$.

Assume $|G|$ is composed with an irrational pencil of curves, we have

$$\begin{aligned}
&K_S + L_m \\
&\geq K_S + \lceil (\pi^*(mL+T) - \frac{1}{\iota_{m_0}} F_{m_0} - S)|_S \rceil \\
&\geq K_S + \lceil (\pi^*(mL+T-m_1L_0) - \frac{1}{\iota_{m_0}} F_{m_0} - S)|_S \rceil + M_{m_1}|_S.
\end{aligned}$$

We can take $Q = (\pi^*(mL + T - m_1L_0) - \frac{1}{\iota_{m_0}} F_{m_0} - S)|_S$ in Lemma 4.4.8 since $\epsilon(m, 0) > 0$. Since $g(C) > 0$, Lemma 4.4.8(ii) implies that Assumption 5.3.2(2) is satisfied for $m \geq m_0 + m_1 + 1$.

We complete the proof. □

In summary, we have a criterion for birationality.

Theorem 5.3.8. *Let (X, L, T) be a polarized triple. Keep the same notation as in Subsection 5.3.1. Then $|K_X + mL + T|$ gives a birational map if*

$$m > \max\left\{m_0 + m_1 + \rho_0 - 1, \mu_0 + m_1 + \frac{2}{\zeta}\right\}.$$

This theorem is optimal in some sense by the following examples.

Example 5.3.9 ([IF00, 14.3 Theorem])**.** Consider the general weighted hypersurface $X_{10} \subset \mathbb{P}(1, 1, 1, 2, 5)$ which is a smooth Calabi–Yau 3-fold. Take $L = \mathcal{O}_X(1)$ and $T = K_X \sim 0$. Then $|5L|$ gives a birational map but $|4L|$ does not.

On the other hand, we may take $m_0 = m_1 = \mu_0 = \rho_0 = 1$ and $\zeta \geq 1$. Hence Theorem 5.3.8 implies that $|mL|$ gives a birational map for all $m \geq 5$.

Example 5.3.10 ([IF00, 14.3 Theorem])**.** Consider the general weighted hypersurface $X_8 \subset \mathbb{P}(1, 1, 1, 1, 4)$ which is a smooth Calabi–Yau 3-fold. Take $L = \mathcal{O}_X(1)$ and $T = K_X \sim 0$. Then $|4L|$ gives a birational map but $|3L|$ does not.

On the other hand, we may take $m_0 = m_1 = \mu_0 = \rho_0 = 1$. Note that $S \in |L|$ and $C \in |L|_S|$. Hence $\zeta = L^3 = 2$. Then Theorem 5.3.8 implies that $|mL|$ gives a birational map for all $m \geq 4$.

Example 5.3.11 ([CCC11, Theorem 4.5])**.** Consider the general weighted complete intersection $X_{2,6} \subset \mathbb{P}(1, 1, 1, 1, 1, 3)$ which is a terminal Calabi–Yau 3-fold. Take $L = \mathcal{O}_X(1)$ and $T = K_X \sim 0$. Then $|3L|$ gives a birational map but $|2L|$ does not.

On the other hand, we may take $m_0 = m_1 = \mu_0 = \rho_0 = 1$. Note that $S \in |L|$ and $C \in |L|_S|$. Hence $\zeta = L^3 = 4$. Then Theorem 5.3.8 implies that $|mL|$ gives a birational map for all $m \geq 3$.

5.4 Birationality on polarized triples

In this section, we consider the birationality problem on polarized triples. By Theorem 5.3.8, we need to estimate m_0, m_1, ρ_0, μ_0, and ζ. First we will give estimation for ρ_0 by Reid's formula. And then we reduce the estimation of m_1 to the estimation of Hilbert polynomial of L so that we can estimate both m_0 and m_1 by Reid's formula. Note that μ_0 can be estimated by Remark 5.3.1 and ζ can be estimated by Proposition 5.3.5 once we have estimation of m_0 and m_1.

We always assume that (X, L, T) is a polarized triple with $\chi(\mathcal{O}_X) > 0$ in this section.

5.4.1 Estimation of ρ_0

In this subsection, we estimate ρ_0. Note that by Theorem 5.1.1(iv)(v) and the fact that $i(X)|I(X)$, we have

$$i(X) \in \{2, 3, 4, 5, 6, 8, 10, 12\}.$$

$i_Q(D)$	0	1	2	3	4	5
$(1,2)$	0	$-1/8$				
$(1,3)$	0	$-2/9$	$-1/9$			
$(1,4)$	0	$-5/16$	$-1/4$	$-1/16$		
$(1,5)$	0	$-2/5$	$-2/5$	$-1/5$	0	
$(2,5)$	0	$-2/5$	$-1/5$	$-1/5$	$-1/5$	
$(1,6)$	0	$-35/72$	$-5/9$	$-3/8$	$-1/9$	$5/72$
$(1,8)$	0	$-21/32$	$-7/8$	$-25/32$	$-1/2$	$-5/32$
$(3,8)$	0	$-21/32$	$-3/8$	$-9/32$	$-1/2$	$-5/32$
$(1,10)$	0	$-33/40$	$-6/5$	$-49/40$	-1	$-5/8$
$(3,10)$	0	$-33/40$	$-3/5$	$-9/40$	$-3/5$	$-5/8$
$(1,12)$	0	$-143/144$	$-55/36$	$-27/16$	$-14/9$	$-175/144$
$(5,12)$	0	$-143/144$	$-19/36$	$-11/16$	$-5/9$	$-31/144$
$i_Q(D)$	6	7	8	9	10	11
$(1,8)$	$1/8$	$7/32$				
$(3,8)$	$-3/8$	$-9/32$				
$(1,10)$	$-1/5$	$7/40$	$2/5$	$3/8$		
$(3,10)$	$-1/5$	$-9/40$	$-3/5$	$-9/40$		
$(1,12)$	$-3/4$	$-35/144$	$2/9$	$9/16$	$25/36$	$77/144$
$(5,12)$	$-3/4$	$-35/144$	$-7/9$	$-7/16$	$-11/36$	$-67/144$

Table A: table of $c_Q(D)$

Since we need to estimate the Hilbert polynomial of some divisor D, we need to estimate the singular part $c_Q(D)$ in Reid's formula. We list all the possible values for $c_Q(D)$ with all the possible singularities in Table A. The first row corresponds to the local index $i_Q(D)$ of D and the first column corresponds to the possible singularities of Q. In the estimation, we will always replace $c_Q(D)$ by the minimal value in the list corresponding to Q.

Note that for a singular point Q of index $r \in \{2,3,4,5\}$ and for any Weil divisor D,

$$c_Q(D) \geq -\frac{r^2-1}{12r}.$$

To estimate ρ_0, we discuss on the value of $i(X)$. Fix a Weil divisor $T' \equiv 0$. Recall that $L^3 \geq \frac{1}{i(X)}$ and $\lambda(L) \geq \frac{1}{i(X)}$.

If $i(X) \in \{2,3,4,5\}$, by Reid's formula and equality (2.5.1),

$$\begin{aligned} h^0(mL+T') &= \chi(\mathcal{O}_X) + \frac{m^3-m}{6}L^3 + m\lambda(L) + \sum_Q c_Q(mL+T') \\ &\geq \chi(\mathcal{O}_X) + \frac{m^3-m}{6}L^3 + m\lambda(L) - \sum_Q \frac{r_Q^2-1}{12r_Q} \\ &= \frac{m^3+5m}{6i(X)} - \chi(\mathcal{O}_X). \end{aligned}$$

Recall that $\chi(\mathcal{O}_X) \leq 4$ (or $\chi(\mathcal{O}_X) = 1$ if $i(X)$=5), hence

$$\rho_0 \leq \begin{cases} 3, & \text{if } i(X) = 5; \\ 4, & \text{if } i(X) \in \{2,3\}; \\ 5, & \text{if } i(X) = 4. \end{cases}$$

If $i(X) = 6$, then we write $B_X = \{a \times (1,2), b \times (1,3), c \times (1,6)\}$. By equality (2.5.1),

$$24\chi(\mathcal{O}_X) = \frac{3}{2}a + \frac{8}{3}b + \frac{35}{6}c.$$

Hence $c < \frac{144}{35}\chi(\mathcal{O}_X)$. By Reid's formula and equality (2.5.1),

$$\begin{aligned} h^0(mL+T') &= \chi(\mathcal{O}_X) + \frac{m^3-m}{6}L^3 + m\lambda(L) + \sum_Q c_Q(mL+T') \\ &\geq \chi(\mathcal{O}_X) + \frac{m^3-m}{6}L^3 + m\lambda(L) - \frac{1}{8}a - \frac{2}{9}b - \frac{5}{9}c \\ &= \frac{m^3-m}{6}L^3 + m\lambda(L) - \chi(\mathcal{O}_X) - \frac{5}{72}c \end{aligned}$$

$$> \frac{m^3+5m}{36} - \frac{9}{7}\chi(\mathcal{O}_X).$$

Recall that $\chi(\mathcal{O}_X) \le 4$, hence $\rho_0 \le 6$.

If $i(X) = 8$, by Morrison [Mor86, Proposition 3], we have $i(X) = I(X)$, $\chi(\mathcal{O}_X) = 1$, and $B_X = \{3 \times (1,2), (1,4), (b_1, 8), (b_2, 8)\}$ for $b_1, b_2 = 1$ or 3. By Reid's formula,

$$\begin{aligned} h^0(mL+T') &\ge 1 + \frac{m^3-m}{6}L^3 + m\lambda(L) + \sum_Q c_Q(mL+T') \\ &\ge 1 + \frac{m^3-m}{6}L^3 + m\lambda(L) - 3 \times \frac{1}{8} - \frac{5}{16} - 2 \times \frac{7}{8} \\ &= \frac{m^3+5m}{48} - \frac{23}{16}. \end{aligned}$$

Hence $\rho_0 \le 4$.

If $i(X) = 10$, by Morrison [Mor86, Proposition 3], we have $i(X) = I(X)$, $\chi(\mathcal{O}_X) = 1$, and $B_X = \{3 \times (1,2), (b_1, 5), (b_2, 5), (c, 10)\}$ for $b_1, b_2 = 1$ or 2, $c = 1$ or 3. By Reid's formula,

$$\begin{aligned} h^0(mL+T') &\ge 1 + \frac{m^3-m}{6}L^3 + m\lambda(L) + \sum_Q c_Q(mL+T') \\ &\ge 1 + \frac{m^3-m}{6}L^3 + m\lambda(L) - 3 \times \frac{1}{8} - 2 \times \frac{2}{5} - \frac{49}{40} \\ &= \frac{m^3+5m}{60} - \frac{7}{5}. \end{aligned}$$

Hence $\rho_0 \le 5$.

If $i(X) = 12$, recall that by Morrison [Mor86, Proposition 3], we have $i(X) = I(X)$, $\chi(\mathcal{O}_X) = 1$, and $B_X = \{2 \times (1,2), 2 \times (1,3), (1,4), (b, 12)\}$ for $b = 1$ or 5. By Reid's formula,

$$\begin{aligned} h^0(mL+T') &\ge 1 + \frac{m^3-m}{6}L^3 + m\lambda(L) + \sum_Q c_Q(mL+T') \\ &\ge 1 + \frac{m^3-m}{6}L^3 + m\lambda(L) - 2 \times \frac{1}{8} - 2 \times \frac{2}{9} - \frac{5}{16} - \frac{27}{16} \\ &= \frac{m^3+5m}{72} - \frac{61}{36}. \end{aligned}$$

Hence $\rho_0 \le 5$.

In summary, we proved the following proposition.

Proposition 5.4.1. *We have the following estimation for ρ_0:*

$$\rho_0 \leq \begin{cases} 3, & \text{if } i(X) = 5; \\ 4, & \text{if } i(X) \in \{2, 3, 8\}; \\ 5, & \text{if } i(X) \in \{4, 10, 12\}; \\ 6, & \text{if } i(X) = 6. \end{cases}$$

5.4.2 Estimation of m_1

Recall that we have a criterion for a linear system not composing with a pencil of surfaces by looking at its Hilbert polynomial by Proposition 2.6.2.

Proposition 5.4.2. *Let L_0 be a nef and big Weil divisor. If*

$$h^0(mL_0) > i(X)L_0^3 m + 1$$

for some integer m, then $|mL_0|$ is not composed with a pencil of surfaces.

5.4.3 Proof of Theorems 1.2.17 and 1.2.18

In this subsection, we prove Theorems 1.2.17 and 1.2.18 by estimating m_0 and m_1.

Proof of Theorem 1.2.17. To prove Theorem 1.2.17, by Section 5.2, we only need to consider polarized triples (X, L, T) with $\chi(\mathcal{O}_X) > 0$. We discuss on the value of $i(X)$. Recall that

$$i(X) \in \{2, 3, 4, 5, 6, 8, 10, 12\}.$$

In the proof, we often use the fact that if Q is a cyclic singular point and D is a Weil divisor with local index $i_Q(D) = 0$, then $c_Q(D) = 0$.

Case 1. $i(X) = 2$ or 3.

In this case, by Reid's formula,

$$h^0(i(X)L) \geq \chi(\mathcal{O}_X) + \frac{i(X)^3}{6}L^3 > 1,$$
$$h^0(2i(X)L) \geq \chi(\mathcal{O}_X) + \frac{8i(X)^3}{6}L^3 > 2i(X)^2L^3 + 1.$$

Hence we can take $L_0 = L$, $m_0 = i(X)$, and $m_1 = 2i(X)$. Then we have $\mu_0 \leq i(X)$ by Remark 5.3.1. By Proposition 5.3.5, $\zeta \geq \frac{1}{i(X)}$. By Proposition 5.4.1, $\rho_0 \leq 4$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 5i(X) + 1$.

Case 2. $i(X) = 4$.

In this case, by the proof of Lemma 2.5.1,

$$\sum_{i=0}^{3} h^0(5L + iK_X) = 4\lambda(5L).$$

Hence there exists i_0 such that

$$h^0(5L + i_0K_X) \geq \lambda(5L).$$

Take $L_0 = L + i_0K_X$. Then

$$\begin{aligned} h^0(5L_0) &= h^0(5L + i_0K_X + 4i_0K_X) \\ &= h^0(5L + i_0K_X) \\ &\geq \lambda(5L) \\ &= 20L_0^3 + 5\lambda(L) \\ &> 5i(X)L_0^3 + 1. \end{aligned}$$

On the other hand,

$$h^0(4L_0) = \chi(\mathcal{O}_X) + \frac{4^3 - 4}{6}L_0^3 + 4\lambda(L) > 4.$$

Hence $h^0(4L_0) \geq 5$ and $|5L_0|$ is not composed with a pencil. Take $m_0 = 4$. By Proposition 5.4.1, $\rho_0 \leq 5$.

If $|4L_0|$ is composed with a pencil, then we have $\mu_0 \leq 1$ by Remark 5.3.1 and we can take $m_1 = 5$. By Proposition 5.3.5, $\zeta \geq \frac{1}{2}$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 14$.

If $|4L_0|$ is not composed with a pencil, then we have $\mu_0 \leq 4$ and we can take $m_1 = 4$. By Proposition 5.3.5, $\zeta \geq \frac{1}{2}$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 13$.

Case 3. $i(X) = 6$.

In this case, recall that $1 \leq \chi(\mathcal{O}_X) \leq 4$ and we write $B_X = \{a \times (1, 2), b \times (1, 3), c \times (1, 6)\}$. By equality (2.5.1),

$$24\chi(\mathcal{O}_X) = \frac{3}{2}a + \frac{8}{3}b + \frac{35}{6}c.$$

If $\chi(\mathcal{O}_X) = 1$, there is only one solution satisfying $i(X) = 6$, which is $B_X = \{5 \times (1, 2), 4 \times (1, 3), (1, 6)\}$. We can take $L_0 = L + i_0K_X$ for some i_0 such that the local index of L_0 at the point $(1, 6)$ is 0. By Reid's formula,

$$h^0(3L_0) \geq 1 + \frac{3^3 - 3}{6}L_0^3 + 3\lambda(L) + \sum_Q c_Q(3L_0)$$

$$\begin{aligned}
&\geq 1+\frac{3^3+15}{36}-5\times\frac{1}{8}\\
&>1,\\
h^0(4L_0)&\geq 1+\frac{4^3-4}{6}L_0^3+4\lambda(L)+\sum_Q c_Q(4L_0)\\
&\geq 1+\frac{4^3+20}{36}-4\times\frac{2}{9}\\
&>2,\\
h^0(7L_0)&\geq 1+\frac{7^3}{6}L_0^3+\sum_Q c_Q(7L_0)\\
&\geq 1+\frac{7^3}{6}L_0^3-5\times\frac{1}{8}-4\times\frac{2}{9}\\
&>7i(X)L_0^3+1.
\end{aligned}$$

Hence $h^0(3L_0)\geq 2$ and $|7L_0|$ is not composed with a pencil. Take $m_0=3$. By Proposition 5.4.1, $\rho_0\leq 6$.

If $|4L_0|$ and $|3L_0|$ are composed with the same pencil, then we have $\mu_0\leq 2$ by Remark 5.3.1. Take $m_1=7$. By Proposition 5.3.5, $\zeta\geq\frac{1}{3}$. By Theorem 5.3.8, $|K_X+mL+T|$ gives a birational map for $m\geq 16$.

If $|4L_0|$ and $|3L_0|$ are not composed with the same pencil, then we can take $m_1=4$ and we have $\mu_0\leq 3$. By Proposition 5.3.5, $\zeta\geq\frac{1}{2}$. By Theorem 5.3.8, $|K_X+mL+T|$ gives a birational map for $m\geq 13$.

Now we assume that $\chi(\mathcal{O}_X)\geq 2$. Note that for any Weil divisor D and singular point Q of index r,

$$c_Q(3D)+c_Q(3D+3K_X)=\begin{cases}-\frac{1}{8}, & \text{if } r=2;\\ 0, & \text{if } r=3;\\ -\frac{3}{8}, & \text{if } r=6.\end{cases}$$

Hence

$$\begin{aligned}
&h^0(3L)+h^0(3L+3K_X)\\
&=2\chi(\mathcal{O}_X)+2\lambda(3L)+\sum_Q(c_Q(3L)+c_Q(3L+3K_X))\\
&=2\chi(\mathcal{O}_X)+2\lambda(3L)-\frac{1}{8}a-\frac{3}{8}c\\
&\geq 2\lambda(3L).
\end{aligned}$$

Therefore there exists a Weil divisor $L_0\equiv L$ such that

$$h^0(3L_0)\geq\lambda(3L)=4L^3+3\lambda(L)>1.$$

On the other hand,

$$h^0(6L_0) \geq \chi + \frac{6^3}{6}L_0^3 > 6i(X)L_0^3 + 1.$$

Hence $|6L_0|$ is not composed with a pencil.

Hence we can take $m_0 = 3$ and $m_1 = 6$. Then we have $\mu_0 \leq 3$ by Remark 5.3.1. By Proposition 5.3.5, $\zeta \geq \frac{1}{3}$. By Proposition 5.4.1, $\rho_0 \leq 6$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 16$.

Case 4. $i(X) = 5$.

In this case, recall that by Morrison [Mor86, Proposition 3], we have $i(X) = I(X)$, $\chi(\mathcal{O}_X) = 1$, and $B_X = \{(b_1, 5), (b_2, 5), (b_3, 5), (b_4, 5), (b_5, 5)\}$ for $b_i = 1$ or 2 for $1 \leq i \leq 5$. We can take $L_0 = L + i_0K_X$ for some i_0 such that the local index of L_0 at the point $(b_1, 5)$ is 0. By Reid's formula,

$$\begin{aligned}
h^0(4L_0) &\geq 1 + \frac{4^3-4}{6}L_0^3 + 4\lambda(L) + \sum_Q c_Q(4L_0)\\
&\geq 1 + \frac{4^3+20}{30} - 4 \times \frac{2}{5}\\
&> 2,\\
h^0(5L_0) &\geq 1 + \frac{5^3-5}{6}L_0^3 + 5\lambda(L)\\
&\geq 1 + \frac{5^3+25}{30}\\
&= 6,\\
h^0(6L_0) &\geq 1 + \frac{6^3-6}{6}L_0^3 + 6\lambda(L) + \sum_Q c_Q(6L_0)\\
&\geq 1 + 35L_0^3 + 6\lambda(L) - 4 \times \frac{2}{5}\\
&> 6i(X)L_0^3 + 1.
\end{aligned}$$

Hence $h^0(4L_0) \geq 3$ and $|6L_0|$ is not composed with a pencil. Take $m_0 = 4$. By Proposition 5.4.1, $\rho_0 \leq 3$.

If $|5L_0|$ and $|4L_0|$ are composed with the same pencil, then we have $\mu_0 \leq 1$ by Remark 5.3.1. Take $m_1 = 6$. By Proposition 5.3.5, $\zeta \geq \frac{2}{5}$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 13$.

If $|4L_0|$ is composed with a pencil, and $|5L_0|$ and $|4L_0|$ are not composed with the same pencil, then we can take $m_1 = 5$ and we have $\mu_0 \leq 2$. By Proposition 5.3.5, $\zeta \geq \frac{2}{5}$. By Theorem 5.3.8, $|K_X+mL+T|$ gives a birational map for $m \geq 13$.

If $|4L_0|$ is not composed with a pencil, then we can take $m_1 = 4$ and we have $\mu_0 \leq 4$. By Proposition 5.3.5, $\zeta \geq \frac{2}{5}$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 14$.

Case 5. $i(X) = 8$.

In this case, recall that by Morrison [Mor86, Proposition 3], we have $i(X) = I(X)$, $\chi(\mathcal{O}_X) = 1$, and $B_X = \{3 \times (1,2), (1,4), (b_1, 8), (b_2, 8)\}$ for $b_1, b_2 = 1$ or 3. We can take $L_0 = L + i_0 K_X$ for some i_0 such that the local index of L_0 at the point $(b_1, 8)$ is 0. By Reid's formula,

$$\begin{aligned}
h^0(4L_0) &\geq 1 + \frac{4^3}{6} L_0^3 + \sum_Q c_Q(4L_0) \\
&\geq 1 + \frac{4^3}{48} - \frac{7}{8} \\
&> 1, \\
h^0(6L_0) &\geq 1 + \frac{6^3}{6} L_0^3 + \sum_Q c_Q(6L_0) \\
&\geq 1 + \frac{6^3}{48} - \frac{5}{16} - \frac{7}{8} \\
&> 4, \\
h^0(8L_0) &\geq 1 + \frac{8^3}{6} L_0^3 + \sum_Q c_Q(8L_0) \\
&= 1 + \frac{8^3}{6} L_0^3 \\
&> 8 i(X) L_0^3 + 1.
\end{aligned}$$

Hence $h^0(4L_0) \geq 2$ and $|8L_0|$ is not composed with a pencil. Take $m_0 = 4$. By Proposition 5.4.1, $\rho_0 \leq 4$.

If $|6L_0|$ and $|4L_0|$ are composed with the same pencil, then we have $\mu_0 \leq \frac{6}{4}$ by Remark 5.3.1. Take $m_1 = 8$. By Proposition 5.3.5, $\zeta \geq \frac{3}{8}$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 16$.

If $|6L_0|$ and $|4L_0|$ are not composed with the same pencil, then we can take $m_1 = 6$ and we have $\mu_0 \leq 4$. By Proposition 5.3.5, $\zeta \geq \frac{3}{8}$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 16$.

Case 6. $i(X) = 10$.

In this case, recall that by Morrison [Mor86, Proposition 3], we have $i(X) = I(X)$, $\chi(\mathcal{O}_X) = 1$, and $B_X = \{3 \times (1,2), (b_1, 5), (b_2, 5), (c, 10)\}$ for $b_1, b_2 = 1$ or 2, $c = 1$ or 3. We can take $L_0 = L + i_0 K_X$ for some i_0 such that the local index of L_0 at the point $(c, 10)$ is 0. By Reid's formula,

$$\begin{aligned}
h^0(4L_0) &= 1 + \frac{4^3-4}{6}L_0^3 + 4\lambda(L) + \sum_Q c_Q(4L_0)\\
&\geq 1 + \frac{4^3+20}{60} - 2\times\frac{2}{5}\\
&> 1,\\
h^0(6L_0) &\geq 1 + \frac{6^3-6}{6}L_0^3 + 6\lambda(L) + \sum_Q c_Q(6L_0)\\
&\geq 1 + \frac{6^3+30}{60} - 2\times\frac{2}{5}\\
&> 4,\\
h^0(8L_0) &\geq 1 + \frac{8^3-8}{6}L_0^3 + 8\lambda(L) + \sum_Q c_Q(8L_0)\\
&\geq 1 + 84L_0^3 + 8\lambda(L) - 2\times\frac{2}{5}\\
&> 8i(X)L_0^3 + 1.
\end{aligned}$$

Hence $h^0(4L_0) \geq 2$ and $|8L_0|$ is not composed with a pencil. Take $m_0 = 4$. By Proposition 5.4.1, $\rho_0 \leq 5$.

If $|6L_0|$ and $|4L_0|$ are composed with the same pencil, then we have $\mu_0 \leq \frac{3}{2}$ by Remark 5.3.1. Take $m_1 = 8$. By Proposition 5.3.5, $\zeta \geq \frac{3}{10}$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 17$.

If $|6L_0|$ and $|4L_0|$ are not composed with the same pencil, then we can take $m_1 = 6$ and we have $\mu_0 \leq 4$. By Proposition 5.3.5, $\zeta \geq \frac{3}{10}$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 17$.

Case 7. $i(X) = 12$.

In this case, recall that by Morrison [Mor86, Proposition 3], we have $i(X) = I(X)$, $\chi(\mathcal{O}_X) = 1$, and $B_X = \{2\times(1,2), 2\times(1,3), (1,4), (b,12)\}$ for $b = 1$ or 5. We can take $L_0 = L + i_0K_X$ for some i_0 such that the local index of L_0 at the point $(b, 12)$ is 0. By Reid's formula,

$$\begin{aligned}
h^0(3L_0) &\geq 1 + \frac{3^3-3}{6}L_0^3 + 3\lambda(L) + \sum_Q c_Q(3L_0)\\
&\geq 1 + \frac{3^3+15}{72} - 2\times\frac{1}{8} - \frac{5}{16}\\
&> 1,\\
h^0(6L_0) &\geq 1 + \frac{6^3-6}{6}L_0^3 + 6\lambda(L) + \sum_Q c_Q(6L_0)
\end{aligned}$$

$$\begin{aligned}
&\geq 1 + \frac{6^3+30}{72} - \frac{5}{16} \\
&> 4, \\
h^0(9L_0) &\geq 1 + \frac{9^3}{6}L_0^3 + \sum_Q c_Q(9L_0) \\
&\geq 1 + \frac{9^3}{6}L_0^3 - 2 \times \frac{1}{8} - \frac{5}{16} \\
&> 9i(X)L_0^3 + 1.
\end{aligned}$$

Hence $h^0(3L_0) \geq 2$ and $|9L_0|$ is not composed with a pencil. Take $m_0 = 3$. By Proposition 5.4.1, $\rho_0 \leq 5$.

If $|6L_0|$ and $|3L_0|$ are composed with the same pencil, then we have $\mu_0 \leq \frac{3}{2}$ by Remark 5.3.1. Take $m_1 = 9$. By Proposition 5.3.5, $\zeta \geq \frac{1}{3}$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 17$.

If $|6L_0|$ and $|3L_0|$ are not composed with the same pencil, then we have $\mu_0 \leq 3$ and we can take $m_1 = 6$. By Proposition 5.3.5, $\zeta \geq \frac{1}{3}$. By Theorem 5.3.8, $|K_X + mL + T|$ gives a birational map for $m \geq 16$. □

Proof of Theorem 1.2.18. Since L has no stable components, take a sufficient divisible k such that $kL \sim M$ is movable and effective and take a sufficient small rational number $\delta > 0$ such that $(X, \delta M)$ is terminal. Run a $(K_X + \delta M)$-MMP with scaling of an ample divisor, it terminates on X' by Kawamata [Kaw92b]. Since $i(X)l(K_X + \delta M) \sim i(X)l\delta M$ is movable for l sufficient divisible, this MMP $\psi : X \dashrightarrow X'$ does not contract any divisors. Hence $(X', \delta\psi_* M)$ is terminal and so is X'. Hence X' is a minimal 3-fold with $K_{X'} \equiv 0$ and $\psi_* L$ is a nef and big Weil divisor by MMP. Note that $|K_X + mL + T|$ gives a birational map if and only if so does $|K_{X'} + m\psi_* L + \psi_* T|$, hence Theorem 1.2.18 follows from Theorem 1.2.17. □

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