§2. Derivative of ideal sheaves.

In this section, we introduce the derivative of ideal sheaves. (Major problem appear in char $p>0$).

Def 2.1 X sm var/k $\boxed{\text{char } 0}$. Let $Der_X : \mathcal{O}_X \to \mathcal{O}_X$ denote the sheaf of k-derivatives, it gives a k-bilinear map

$$Der_X \times \mathcal{O}_X \to \mathcal{O}_X.$$

$D(I) := Im(Der_X \times I)$

In local coordinates near $p:(x_1, \cdots, x_n)$, I generated by $f_1 \cdots f_s$

$$D(I)_p = \left\{ \frac{\partial g}{\partial x_i} \mid g \in I \right\} \overset{☆}{=} \left(\frac{\partial f_i}{\partial x_j}, f_i \mid 1 \le i \le s,\ 1 \le j \le n \right)$$

Rem: ($f = \frac{\partial(xf)}{\partial x} - x\frac{\partial f}{\partial x}$).

and we def

$D^{r+1}(I) = D(D^r(I))$. $\quad I \subset D(I) \subset \cdots \subset D^{m-1}(I) \subset D^m(I) = \mathcal{O}_X$ $\quad m = \max \operatorname{ord} I$,

Obviously $D^r(D^s(I)) = D^{r+s}(I)$.

For marked ideals (I, m), $D^r(I, m) = (D^r(I), m-r)$.

Rem: ① In char $p>0$, the correct derivative is $\frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial u^{|\alpha|}} = D^\alpha$.

and (try to def:

$$D^\alpha(I) = \left(\frac{\partial^{|\beta|}}{\partial u^{|\beta|}} f_j \quad 0 \le |\beta| \le |\alpha|,\ f_j \text{ generator of } I,\ u \text{ local cor} \right).$$

② In this case, $D^i(D^j(I)) \nsubseteq D^{i+j}(I)$ might happen.

for: char $p=2$ $\quad I = (x^3)$.

$D^1(D^1(I)) = D^1(x^3, 3x^2) = (x^3, 3x^2) \not\supset$

$D^2(I) = (x^3, \frac{\partial x^3}{\partial x}, \frac{1}{2}\frac{\partial^2 x^3}{\partial x^2}) = (x^3, 3x^2, 3x)$

Prop 2.2 Notations as above.

① $D^r(I\cdot J) \subset \sum_{i\geq 0}^{r} D^i(I)\, D^{r-i}(J)$

② $\mathrm{Supp}(I, m) = \mathrm{Supp}(D^r(I), m-r)$ for $r < m$ char 0.

③ $h: Y \to X$ sm, then $D(h^{-1}I\cdot \mathcal{O}_Y) = h^{-1}D(I)\cdot \mathcal{O}_Y$

Proof: ① follows from chain rule.

② local set $I \in (f_1 \cdots f_s)$ $D(I) = (f_i, \frac{\partial f_i}{\partial x_j})$ locally. near x

if $x \in \mathrm{Supp}(I, m) \Rightarrow \mathrm{ord}_x f_i \geq m \Rightarrow \mathrm{ord}_x \frac{\partial f_i}{\partial x_j} \geq m-1$.

if $x \in \mathrm{Supp}(D(I), m-1) \Rightarrow \mathrm{ord}_x \frac{\partial f_i}{\partial x_j} \geq m-1 \Rightarrow$ ✓.

$\mathrm{Supp}(I, m) = \mathrm{Supp}(D(I), m-1)$, inductively we are done.

③
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h"'] & X\times A^n \arrow[dl, "\pi"] \\ X & \end{tikzcd}$$
I on $X\times A^n$, J on X.

g etale, π proj.

$D(\pi^{-1}J\cdot \mathcal{O}_{X\times A^n}) = \pi^{-1}D(J)\cdot \mathcal{O}_{X\times A^n}$. ✓.

now we check etale.

Now we consider $I_p \otimes \hat{\mathcal{O}}_p$, we have.

$\forall\ y \in Y,\ x = g(y)$, $\widehat{\mathcal{O}_{Y,y}} = \widehat{\mathcal{O}}_{X\times A^n, x}$

$\Rightarrow$ Commutative follows.

□.

Remark: for (1), Set $I = (f)$, $J = (g)$ $IJ = (fg)$

$D(IJ) = (fg, \partial(fg))$

$\subsetneq$

$D(I)J + ID(J) = (fg, f\partial g, (\partial f)\cdot g)$.

Lemma 2.3 (Bir transform and derivative ideal)
Let (I, m) be a marked ideal, $\pi: Y \to X \supset Z$ a smooth blow up with center $Z \subseteq \mathrm{Supp}(I, m)$

Then $\pi_*^{-1}(D^j(I, m)) \subset D^j(\pi_*^{-1}(I, m))$ for $j \geq 0$.

Proof. This is a local problem, take $y \in Y$, $x \in Z \subset X$. choose local chart $(x_1 \cdots x_n)$ ~~near~~ x s.t. $Z = (x_1 = \cdots = x_r = 0)$

and the local chart resp to x_r on $Bl_Z X$:

$$y_1 = x_1/x_r, \cdots, y_{r-1} = x_{r-1}/x_r, \quad y_r = x_r, \cdots, y_n = x_n.$$

$\forall f \in I$ $\quad \pi_*^{-1}(f, m) = y_r^{-m} f(y_1 y_r, \cdots, y_{r-1} y_r, y_r, \cdots y_n)$

$$\begin{cases} \pi_*^{-1}\left(\frac{\partial f}{\partial x_r}, m-1\right) = y_r \frac{\partial}{\partial y_r} \pi_*^{-1}(f, m) - y_r \sum_{i<r} \frac{\partial}{\partial y_i} \pi_*^{-1}(f, m) + \boxed{(m-1) \cdot \pi_*^{-1}(f, m)} & j = r \\ \pi_*^{-1}\left(\frac{\partial}{\partial x_j} f, m-1\right) = \frac{\partial}{\partial y_j} \pi_*^{-1}(f, m) \cdot y_r & j > r \\ \pi_*^{-1}\left(\frac{\partial}{\partial x_j} f, m-1\right) = \frac{\partial}{\partial y_j} \pi_*^{-1}(f, m) & j < r. \end{cases}$$

product of marked ideal

$\Rightarrow \quad \pi_*^{-1}(D(I, m)) \subset D(\pi_*^{-1}(I, m))$

Inductively we are done.

A major idea of Hironaka is that, instead of dealing with I, we deal with some "equivalent ideal" that enrich I, and the enriched ideal behaves well under centern restriction.

Def 2.4 (Coefficient ideal and Homogenized ideal)
Let (I, m) be a marked ideal such that $m = \text{max-ord}\, I$ on sm var $X / \mathrm{char}\, p = 0$ k.

We def $\quad$ D-Balanced: $(D^i I)^m \subset I^{m-i}$ $\forall i < m$ $\quad W(I, n)$.

$$\boxed{C(I, m)} = (I, m) + D(I, m) + \cdots + D^{m-1}(I, m). \quad (+ \cdots + D^\infty(I, m))$$

and $\quad$ MC-Invariant: $T(I) \cdot D(I) \subset I$

$$\boxed{H(I, m)} = (H(I), m) = (I, m) + D(I, m) \cdot (T(I), 1) + \cdots D^i(I, m) \cdot (T(I), 1)^i + \cdots + D^{m-1}(I, m) \cdot (T(I), 1)^{m-1}$$

$$= (I + DI \cdot TI + \cdots + D^i I \cdot (TI)^i + \cdots + \underbrace{D^{m-1} I \cdot (TI)^{m-1}}, m).$$

Here $\underbrace{T(I) = D^{m-1} I}$. ☆

$x^2 + y^3 \qquad C(I) = (x^2 + y^3, 2) + ((x, y^2), 1)^2$

$$= \underbrace{(x^2, xy^2, y^3, 2)}.$$

[Wat 05]

Prop 2.5 (1) $\mathrm{Supp}(\mathcal{H}(I,m)) = \mathrm{Supp}(C(I,m)) = \mathrm{Supp}(I,m)$.

(2) $\forall Z \subseteq \mathrm{Supp}((I,m))$ smooth on X, $\pi : Bl_Z X \to X$, we have

$\mathrm{Supp}(\pi_*^{-1} \mathcal{H}(I,m)) = \mathrm{Supp}(\pi_*^{-1} C(I,m)) = \mathrm{Supp}(\pi_*^{-1}(I,m))$

(3) $h : Y \to X$ smooth, then

$\mathcal{H}(h^{-1}I \cdot \mathcal{O}_Y) = h^{-1}\mathcal{H}(I) \cdot \mathcal{O}_Y$

$C(h^{-1}I \cdot \mathcal{O}_Y, m) = h^{-1}C(I.m) \cdot \mathcal{O}_Y$.

Proof:

(1) By Def-Prop 1.4 (1)-(3) Prop 2.2 (2)

$$\underset{\cong}{\mathrm{Supp}(\mathcal{H}(I.m))} = \bigcap_{i=0}^{m-1} \mathrm{Supp}\left(D^i(I.m) \cdot (T(I),1)^i\right) \supseteq \bigcap_{i=0}^{m-1} \underbrace{\mathrm{Supp}(D^i(I.m))}_{= \mathrm{Supp}(I.m)} \cap \mathrm{Supp}(T(I),1)$$

$\mathrm{Supp}(I.m)$

Similar for $C(I.m)$.

(2) $\mathrm{Supp}(\pi_*^{-1} \mathcal{H}(I.m)) = \bigcap_{i=0}^{m-1} (\pi_*^{-1} D^i(I.m) \cdot \pi_*^{-1}(T(I)^i, i))$

$\cong \mathrm{Supp}(\pi_*^{-1} I.m)$

Lem2.3 $\cup$ $\bigcap_{i=0}^{m-1} \mathrm{Supp}(D^i(\pi_*^{-1}I, m-i) \cdot T(\pi_*^{-1}I)^i, i)$

$= \mathrm{Supp}(\pi_*^{-1} I, m)$.

Similar for $C(I.m)$.

(3) Follows from Prop 2.2 (3).

Rem: Above properistion says that, any order reduction process

To be more specific

$$\begin{array}{cccccc} \pi : X_r \to & X_{r-1} & \to \cdots \to & X_0 & = X & \text{a seq of blow-up} \\ & \cup & & \cup & & \\ & Z_{r-1} & & Z_0 & & \end{array}$$

(1) $Z_i \subseteq \mathrm{Supp}(I_i, m)$ iff $Z_i \subseteq \mathrm{Supp}\,\mathcal{H}(I_i, m)$, $(C(I_i, m))$

(2) $\mathrm{Supp}((I, m)_r) = \phi$ iff $\mathrm{Supp}(\mathcal{H}(I.m))_r = \phi$ $(C(I.m)_r)$.

$\S$ bir trans on X_r.

(3) Prop 2.5 (3) gaurantees sm functoriality for $\mathcal{H}(I)$, $C(I)$, vice versa.

Now we consider the restriction problem

Prop 2.6 Let (X, I, m) be triple s.t. (I, m) marked ideal on sm $X/k = 0$
S smooth subvariety on X not contained in $\mathrm{Supp}(I, m)$, $Z \subseteq S \cap \mathrm{Supp}(I, m)$
$\pi: Bl_Z X \to X$ the smooth blow up, $\pi|_S : Bl_Z S \to S$

$$\underset{S'}{\overset{\|}{Bl_Z S}}$$

Then (1) $\mathrm{Supp}(I, m) \cap S \subseteq \mathrm{Supp}(I|_S, m)$

(2) $\mathrm{Supp}(C(I, m)) \cap S = \mathrm{Supp}(C(I, m)|_S)$

(3) $\pi|_{S*}^{-1}((I, m)|_S) = (\pi_*^{-1}(I, m))|_{S'}$

(4) $\mathrm{Supp}(\pi_*^{-1} C(I, m)) \cap S' = \mathrm{Supp}(\pi|_{S*}^{-1}(C(I, m)|_S))$

Proof: (1) follows from the fact that when we do restriction, ord will not decrease.

(2) Let $x_1 \cdots x_k, y_1 \cdots y_{n-k}$ be local parameters at x s.t. $x \in S$
$S := (x_1 = \cdots = x_k = 0)$ $\forall f \in I$, $f = \sum C_{\alpha, f} x^\alpha = \sum C_{\alpha, f}(y) x^\alpha$

Now, $x \in \mathrm{Supp}(I, m) \cap S$ iff $\mathrm{ord}_x (C_{\alpha, f}(y))|_S \geq m - |\alpha|$ for all $f \in I$
$1 \leq |\alpha| \leq m-1$

$$C_{\alpha, f}|_S = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}\Big|_S \in D^\alpha(I)|_S.$$

thus

$$\mathrm{Supp}(I, m) \cap S = \bigcap_{\substack{f \in I \\ |\alpha| \leq m}} \mathrm{Supp}(C_{\alpha, f}|_S, m - |\alpha|) \supseteq \bigcap_{0 \leq i \leq m-1} \mathrm{Supp}(D^i I|_S, m-i)$$

$$\mathrm{Supp}(C(I, m)) = \mathrm{Supp}(I, m) \cap S \supseteq \mathrm{Supp}(C(I, m)|_S).$$

$$= \mathrm{Supp}(C(I, m)|_S).$$

(3). Notations as in (2), $Z \subset S \subset X$, $\pi|_S : S' \to S$.
$Z := (x_1 = \cdots = x_k = y_1 = \cdots = y_q = 0)$.

for $x \in Z \subset S \subset X$, locally blow up can write as

$$x_1' = x_1/y_q \cdots x_k' = x_k/y_q, \; y_1' = y_1/y_q \cdots y_q' = y_q \; y_{q+1}' = y_{q+1} \cdots$$

then strict transform of S (denoted as S') is locally defined by
$x_1' = x_2' = \cdots = x_k' = 0 \subset X' = Bl_Z X$.

for $f \in I$ locally $f = \sum C_{\alpha,f}(y)\, x^{\alpha} \Rightarrow \pi_*^{-1}(f.m) = \boxed{\sum C'_{\alpha,f}(y)\, \underset{\sim}{x'^{\alpha}}}$

where

$$C'_{\alpha,f}(y') = y_q^{|\alpha|-m}\, C_{\alpha,f}(y_1' y_q', \dots, y_q', \dots, y_m')$$

while

$$f|_S = C_{0,f}|_S \quad \text{and} \quad \pi_*^{-1}(f.m)|_{S'} = C'_{0,f}(y')|_{S'},$$

$$\pi_*^{-1}(f.m)|_{S'} = (C'_{0,f})|_{S'} = y_q^{-m}\,(\underbrace{\pi^* C_{0,f}}_{\text{Composition.}})|_{S'} = y_q^{-m}\, \pi|_S^*(C_{0,f}|_S)$$

$$= y_q^{-m}\, \pi|_S^*(f|_S) = \pi|_{S*}^{-1}(f|_S.m)$$

$$\Rightarrow \quad \pi_*^{-1}(I.m)|_{S'} = \pi|_{S*}^{-1}((I.m)|_S).$$

(4). Notations as in (3).

As before, $* \; \mathrm{Supp}(\pi_*^{-1} I, m) \cap S' = \bigcap_{\substack{f \in I \\ |\alpha| < m}} \mathrm{Supp}(C'_{\alpha,f}|_{S'}, m - |\alpha|)$

Now $C'_{\alpha,f} = y_m^{-m+|\alpha|}\, \pi^*(C_{\alpha,f}) \subseteq \pi_*^{-1} D^{|\alpha|}(I.m)$.

back to $*$ LHS $\supseteq \bigcap_{0 \le i < m} \mathrm{Supp}(\pi_*^{-1} D^i(I.m)|_{S'})$

$$= \bigcap_{i=0} \mathrm{Supp}(\pi|_{S*}^{-1}(D^i(I.m)|_S)).$$

$$= \mathrm{Supp}(\pi|_{S*}^{-1} C(I.m)|_S).$$

So $S' \cap \mathrm{Supp}(\pi_*^{-1} C(I.m)) \supseteq S' \cap \mathrm{Supp}(\pi_*^{-1}(I.m)) \supseteq \mathrm{Supp}(\pi|_{S*}^{-1} C(I.m)|_S)$.

$\mathrel{\rotatebox{90}{$=$}}$

$\mathrm{Supp}(\pi_*^{-1} C(I.m)|_{S'})$

$\mathrel{\rotatebox{90}{$=$}}$

$\mathrm{Supp}(\pi|_{S*}^{-1}(C(I.m)|_S))$.

$\square$.

Remark: The above proposition says that, $S \subset X$ not contained in $supp(I,m)$, an order reduction for $C(I,m)|_S$ on S lifts naturally to an "order reduction" on X.

To be more specific $\quad Z \subset S \subset X$

$$\begin{tikzcd}
\mathbb{B}: & \pi: & X_r \arrow[r] & \cdots \arrow[r] & X_0 & C(I,m). \\
\mathbb{B}_S \arrow[u, "\text{lift}"] : & \pi_S: & S_r \arrow[u] \arrow[r] & \cdots \arrow[r] & S_0 \arrow[u] & \supseteq C(I,m)|_S
\end{tikzcd}$$

(1) $Z_i \subseteq Supp([C(I,m)|_S]_i) \Rightarrow Z_i \subseteq Supp[C(I,m)_i] \cap S_i$

(2) $Supp([C(I,m)|_S]_r) = \emptyset \Rightarrow Supp[C(I,m)_r] \cap S_r = \emptyset$.

i.e. $supp(I,m)_r = supp(C(I),m)_r$ is disjoint with S_r.

(3) If $\mathbb{B}_S$ is functorial resp to smooth morphisms, then the natural lifting is also functorial resp to smooth morphism.

In fact
$$\begin{tikzcd}
& S_Y \arrow[r] \arrow[d, "h_Y"'] & Y \arrow[d, "h \to \text{smooth}"] \\
Z \subset & S \arrow[r, hook] & X
\end{tikzcd}$$

$h: Y \to X$ smooth $\Rightarrow h_Y: S_Y \to S$ smooth.

Lift $\mathbb{B}_S$ to $\mathbb{B}$, blow-up center is Z.

functoriality for $\mathbb{B}_S$ imply blow up center for S_Y is $h_Y^{-1}(Z)$.

lift ~~to~~ Y, blow up center is again $h_Y^{-1}(Z) \subset Y$.

$\square$

Rem: In previous case, we only consider the restriction $\to$ ord reduction $\to$ lifting that end up with $S_r \cap Supp(I,m)_r = \emptyset$.

Key: If we can find $S \supseteq Supp(I,m)$ such that each time.

maxi cont $\longrightarrow$ $S_i \supseteq Supp[(I,m)_i]$ then we end up with

$\emptyset = S_r \cap Supp(I,m)_r = Supp(I,m)_r$!

Def-Prop 2.7 (Hypersurface of Maximal contact).

The maxi contact ideal sheaf of (I,m) is $T(I) := D^{m-1}(I,m)$ $\quad m = maxord\, I$.

For any $x \in Supp(I,m) = Supp(T(I))$, $\exists$ open neighbor $x \in U_x$, and a smooth element $h \in T(I)(U)$ ($V(h) =: H$ is sm hypersurface on U_x). with $I|_H \neq 0$, we call H a hypersurface of maximal contact.

Rem: x^2+y^3, $maxord = 2$, $D((x^2+y^3)) = (x, y^2)$ $\quad x + cy^2$ is a hy surf of m.c.

Now, $\pi: \underset{U_1}{\underset{\|}{Bl_Z U}} \longrightarrow U$ a sm blow up with $Z \subseteq Supp(I.m) \cap H$, we have

$Supp(\pi_*^{-1}(I.m)) \subset \pi^{-1}H$..

Proof: $\underset{\underset{Supp(I\cdot m)}{\|}}{Supp(T(I),1)} \subseteq V(h) = H$.

since $\pi_*^{-1}(h,1) \subseteq \pi_*^{-1}(T(I),1) \subseteq (T(\pi_*^{-1}(I)),1)$

$\Rightarrow Supp(\pi_*^{-1}I, m) = Supp(T(\pi_*^{-1}I),1) \subseteq supp(\pi_*^{-1}h.)) = h^{-1}H$.

Rem! the maximal contact hypersurface is local and depends on choice of h.

(That is where $\mathcal{H}(I)$ plays a role).

Lem 2.8 Let $(X, I, m\ E)$ be a marked triple, $m = \max ord\ I$.

for any $u, v \in T(X, m)_x$ at $x \in Supp(I.m)$ that are smooth and snc with E. Then we have automorphism

$\widehat{\Phi}_{uv}$ of $\widehat{X}_x = Spec\ \widehat{O}_{x,X}$

s.t. (1) $\widehat{\phi}^*_{uv}(\mathcal{H}(I))_x = (\widehat{\mathcal{H}I})_x$

(2) $\widehat{\phi}^*_{uv} E = E$

(3) $\widehat{\phi}^*_{uv}(u) = v$

(4) $Supp(\hat{I}.m) = V(T(\hat{I},m))$ is in the fixed point set of $\widehat{\phi}_{uv}$.

Proof: Step 1 construction.

Take $u = u_1, u_2, \cdots, u_n$ s.t. both u or v, $u_2, u_3 \cdots u_n$ form local coordinates and is compatible with E.

Set $\widehat{\Phi}_{uv}(u) = v$ $\quad \widehat{\Phi}_{uv}(u_i) = u_i$ for $i > 0$.

Step 2: Varification.

Let $h := v - u \in T(I)$. $\forall f \in \widehat{I}$.

$$\begin{aligned}\widehat{\phi}^*_{uv} f &= f(u_1 + h, u_2, \cdots, u_n) \\ &= f(u_1, \cdots, u_n) + \frac{\partial f}{\partial u_1} h + \frac{1}{2!}\frac{\partial^2 f}{\partial u_1} h^2 + \cdots \\ &\subseteq \widehat{I} + \widehat{DI}\cdot\widehat{TI} + \cdots + \underset{\frac{\partial^i f}{\partial u_i}}{\underset{\Psi}{\widehat{D^i I}}} \cdot \underset{h^i}{\underset{\Psi}{\widehat{TI}^i}}\end{aligned}$$

$\Rightarrow \widehat{\phi}^*_{uv} \hat{I} \subset H\hat{I}$. Similarly $\widehat{\phi}^*_{uv}(D^i\hat{I}) \subset \mathcal{H} D^i\hat{I}$ $\quad \widehat{\phi}^*_{uv} \widehat{T(I)} \subset \underset{\widehat{T(I)}}{\underset{\Downarrow}{\mathcal{H}(\widehat{T(I)})}}$

to sum up, $\hat{\phi}^*_{uv}(D^i \hat{I} \cdot \hat{T}(I)^i) \subset D^i \hat{I} \cdot \hat{T}(I)^i + \cdots + D^{m-1} \hat{I} \cdot \hat{T}(I)^{m-i-1} \cdot \hat{T}(I)^{m_i} \subset H(\hat{I})$.

$\Rightarrow \quad \hat{\phi}^*_{uv} \mathcal{H}(\hat{I}) \subset \mathcal{H}(\hat{I})$. Noetherian properties garrantees that
$\hat{\phi}^{*\,n}_{uv}(\mathcal{H}(\hat{I})) = \hat{\phi}^{*\,n-1}_{uv}(\mathcal{H}(\hat{I})) \Rightarrow (1)\ \checkmark$.

(2) (3) $\checkmark$ by construction

(4) $h=0$ is fixed by $\hat{\phi}_{uv}$ $\Rightarrow$ $\mathrm{Supp}(T(I), 1)$ is fixed $\Rightarrow$ $\mathrm{Supp}(I, m)$ fixed.

Formal local uniqueness imply étale equivalence.

Lem 2.9 Settings as in Lem 2.8.
Then there exists étale neighborhoods

$$\begin{tikzcd} \tilde{x} \in U \arrow[r, bend left, "\phi_u"] \arrow[r, bend right, "\phi_v"'] & X \ni x \end{tikzcd}$$

$\phi_u, \phi_v : \underset{\tilde{x}}{U} \to \underset{x}{X}$ of $x = \phi_u(\tilde{x}) = \phi_v(\tilde{x})$

s.t. (1) $\phi_u^*(X, \mathcal{H}(I), m, E) = \phi_v^*(X, \mathcal{H}(I), m, E) := (\tilde{X}, \mathcal{H}(\tilde{I}), m, \tilde{E})$

(2) $\phi_u^*(u) = \phi_v^*(v)$

(3) $\mathbb{B}: X_r \to \cdots \to X_0$ be a seq of sm blow-up with Z_i in $\mathrm{supp}(I, m)$

then $\phi_u^* \mathbb{B}(X, \mathcal{H}(I), m, E) = \phi_v^* \mathbb{B}(X, \mathcal{H}(I), m, E) : \tilde{X}_r \to \cdots \to \tilde{X}_0$

$\phi_{ui}, \phi_{vi} : \tilde{X}_i \to X_i$ satisfies

$\phi_{ui}^{-1}(V(u), i) = \phi_{vi}^{-1}(V(v), i)$ and

$\phi_{ui}(\tilde{y}_i) = \phi_{vi}(\tilde{y}_i) \quad \forall \tilde{y} \in \mathrm{Supp}(\tilde{I}_i, m)$.

Remark: Lemma 2.9 allow us to glue restricted resolution! $\forall x \in X$.
that is, $\forall\, U(u, x)$ and $U(v, x)$ two open set that restrict to $V(u)$, $V(v)$
and def blow up seq and lift to Blow up seq $B_u(U(u,x))$ $B_v(U(v,x))$

$$\exists \quad \begin{tikzcd} U(uv, x) \arrow[r, bend left, "\phi_u"] \arrow[r, bend right, "\phi_v"'] & U(u,x) \cap U(v,x). \end{tikzcd}$$

s.t. $\phi_u^* B(U_u \cap U_v) = \phi_v^* B(U_u \cap U_v)$.

$\Rightarrow$ restricted to $U_u \cap U_v$, the blow up center for $B_u(U)$, $B_v(U)$ coincide!
we can glue blow up center and globalize it as in <u>LL</u>.
and sm func preserved.