Boundedness & volume of generalised pairs

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§1. Introduction.

§1.1 Background.

Thm (HMX14) (DCC of volume)

$d \in \mathbb{Z}_{>0}$, $\Phi \subseteq [0,1]$ DCC set

$$\left\{ \operatorname{vol}(K_X + B) \;\middle|\; \begin{array}{l} (X, B) \text{ lc} \quad \dim X = d \\ B \in \Phi \end{array} \right\} : \text{DCC}$$

Thm (HMX'18)

$d \in \mathbb{Z}_{>0}$ & $\Phi \subseteq [0,1]$ DCC, $v \in \mathbb{Q}_{>0}$

$$\left\{ X \;\middle|\; \begin{array}{l} (X,B) \text{ lc} \quad \dim X = d \\ B \in \Phi \quad K_X+B \text{ ample} \quad \mathrm{vol}(K_X+B)=v \end{array} \right\} \text{ bdd.}$$

Goal. gen to generalised pairs & app

§1.2. Main result.

Notation $d \in \mathbb{Z}_{>0}$, $\Phi \subseteq \mathbb{R}^{\geq 0}$ DCC set, $v \in \mathbb{R}_{>0}$

with data $X' \xrightarrow{f} X$

M' nef on X', $M = f_* M'$

$$\mathcal{G}_{glc}(d, \Phi) = \left\{ (X, B+M) \;\middle|\; \begin{array}{l} \dim X = d \\ (X, B+M) \text{ g-pair lc} \\ B \in \Phi \\ M' = \sum \mu_i M_i', \quad M_i' \text{ nef Cartier div on } X' \\ \mu_i \in \Phi \end{array} \right\}$$

(under B and M: $\triangle$, $\triangle$)

NQC-div (Han–Li)

$$\mathcal{G}_{glc}(d, \Phi, \leq v) = \left\{ (X, B+M) \in \mathcal{G}_{glc}(d, \Phi) \;\middle|\; \mathrm{vol}(K_X+B+M) \leq v \right\}$$

(variants written above: $<v$, $=v$ in place of $\leq v$; and "K_X+B+M big")

Def. (g-pair)

Assume $X' \to X$ log resol of (X, B)

$(X, B+M)$ $\quad X' \xrightarrow{f} X$, M' nef $\mathbb{R}$-div on X' s.t.

$f_* M' = M$ & $K_X + B + M$ $\mathbb{R}$-Cartier

Similar defn singularities of $(X, B+M)$.

$$f^*(K_X + B + M) = K_{X'} + B' + M'. \quad \text{for } \exists B'$$

$\forall$ prime $E \subseteq X'$ def log discrepancy

$$a(E, X, B+M) = 1 - \operatorname{mult}_E B'$$

$(X, B+M)$ klt (resp. lc) iff $\forall E$ $\quad a(E, X, B+M) > 0$ (≥ 0)

if $M' = 0$. ($M' \equiv 0 / X$) def $\Longleftrightarrow$ usual def of usual pairs.

(
① subadj. $\quad K_X + \Delta|_F \sim K_F + \Theta_F + P_F \leftarrow$ moduli pair

② cbf $\quad K_X + \Delta \sim f^*(K_Z + B_Z + M_Z)$
)

⊗ Descent of nef div

Thm A d, Φ, v — bdd family (couples)

$\exists \mathfrak{P} = \mathfrak{P}(d, \Phi, v)$ s.t. $\forall (X, B+M) \in \mathcal{G}_{glc}(d, \Phi, < v)$

$\exists$ log sm couple $(\bar{X}, \bar{\Sigma}) \in \mathfrak{P}$ & $\bar{X} \dashrightarrow X$ s.t.
① $\mathrm{Supp}\, \bar{\Sigma} \supseteq \mathrm{Ex}(\bar{X} \dashrightarrow X) \cup$ st. transf of B
② M_i descends to $\bar{X}$

⊗ DCC of volumes

Thm B. $\{ \mathrm{vol}(K_X + B + M) \mid (X, B+M) \in \mathcal{G}_{glc}(d, \Phi) \}$: DCC

⊗ Bddness

Thm C. $\mathcal{F}_{gklt}(d, \Phi, v) = \{ (X, B+M) \in \mathcal{G}_{glc}(d, \Phi, v),\ K_X + B + M \text{ ample} \}$ bdd family.

⊗ DCC lit. vol (conj by Zhan. Li)

Thm D. (tbc).

Thm A d, Φ, v — bdd family (couples)

$\exists \mathfrak{P} = \mathfrak{P}(d, \Phi, v)$ s.t. $\forall (X, B+M) \in \underline{\mathcal{G}_{glc}(d, \Phi, <v)}$

$\exists$ log sm couple $(\bar{X}, \bar{\Sigma}) \in \mathfrak{P}$ & $\bar{X} \dashrightarrow X$ s.t.
① $\mathrm{Supp}\,\bar{\Sigma} \supseteq \mathrm{Ex}(\bar{X} \dashrightarrow X) \cup$ st. transf of B
② M_i descends to $\bar{X}$

Descend. M nef Cart on X; p, q: $W \to X$, $W \to Y$ ($X \dashrightarrow Y$ birat'l map).
We say M descends to Y as L if L s.t. $p^* M = q^* L$.

Idea Const birat'l bdd family & use Acc for g-lct

Example. $p \in \mathbb{Z}_{>0}$ & (X, B) klt & $(X, B+M)$ glc. $pB \in \mathbb{Z}$, pM Cart nef

$(X_0, B_0 + M_0) = (X, B+M)$ $\quad \lambda_0 = \mathrm{glct}(X_0, B_0; M_0)$

lc $(X_0, B_0 + \lambda_0 M)$

$f_1 \uparrow$ dlt

$(X_1, \bar{B}_0 + \lambda_0 M_1) \longleftarrow \underline{(X_1, B_1 + \lambda_0 M_1)}$, $B_1 = \bar{B}_0 - \frac{1}{p}\lfloor \bar{B}_0 \rfloor$ klt

$\exists \delta = \delta(d, p, r)$ s.t. $B_0 - \delta \lfloor B_0 \rfloor \geq 0$.

$$
\begin{array}{ccc}
 & & (X_1, \bar{B}_1 + \lambda_1 M_1) \\
 & & \uparrow f_2 \ \text{dlt} \\
\bar{B}_1 - \frac{1}{p}\lfloor \bar{B}_1 \rfloor = B_2 & & \\
(X_2, B_2 + \lambda_2 M_2) & \rightsquigarrow & (X_2, \bar{B}_1 + \lambda_1 M_2)
\end{array}
$$

le $\lambda_1 = \mathrm{glct}(X_1, \bar{B}_1; M_1) > \lambda_0$

$\lambda_2 = \mathrm{glct}(X_2, B_2; M_2) > \lambda_1 > \lambda_0$

$\uparrow$ dlt

$\cdots\cdots$ terminate $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$

contr glct Acc

$\exists\, l$ s.t. $\lambda_l = +\infty$. (M descend to X_l as M_l).

① glct v.s. descend of nef.

② Construct model

Lem 1. $X \xrightarrow{f} Y$ (birat'l) Fano type (X,B) lc & M nef Cartier on X

s.t. $\exists\, u > 2d$, $-(K_X+B+uM)$ nef $/Y$. $\Rightarrow$ M descends to Y.

($M = f^* f_* M$)

Proof klt (X,Δ) & $-(K_X+\Delta)$ ample $/Y$

$\Rightarrow$ $X \to Y$ contraction of an ext'l face of Mori-Kleiman cone of X.

$\Leftrightarrow$ $M \equiv 0 / Y$.

$$\overline{NE}(X/Y)_{(K_X+\Delta)<0} = \overline{NE}(X/Y) = \overline{NE}(X)_{f^*H=0}$$

$\forall R$: ext'l ray, $\exists\, C$ ext'l curve s.t.

$\exists\, G$, $-(K_X+B+M)$ semiample

(<0)

$-(K_X+B)\cdot C < 2d$

$\Rightarrow K_X+B+G+M \equiv 0/Y$

$-(K_X+B+uM)\cdot C \geq 0 \Rightarrow uM\cdot C \leq -(K_X+B)\cdot C < 2d$

$\Rightarrow$ $M \cdot C = 0$.

$M \equiv 0 / Y$. ∎

$X \dashrightarrow \cdots \dashrightarrow X_l$, $\pi: X \to Y$, $X_l = Y$

Birkar

Rem MMP. ✓

Lem 2. $d, p \in \mathbb{Z}_{>0}$ $(X, B+M)$ d-dim'l. pM' Cartier

(X,B) klt, $\lambda = \mathrm{lct}(X,B;M)$. TFAE

1) $\lambda \geq 3dp$

2) $\lambda = +\infty$

3) M' descends to X & pM' Cartier ref.

Proof. 3) ⇒ 2) ⇒ 1) ✓.

1) ⇒ 3). $K_{X'}+B'+3dpM'$ MMP/X

(orange note: each step M' descends. ⇝ M' desc'd X'')

$X' \dashrightarrow X''$ (via ϕ), $f: X' \to X$, $f'': X'' \to X$, dlt

$f''^*(K_X+B) = K_{X''}+B''$

Claim $X'' \to X$: Fano type. M' descends to X by lem 1.

$B' = f_*^{-1}B + Ex(f)$ $+\lambda M$

(X,B) klt

$K_{X'}+B' = f^*(K_X+B+\lambda M) + (\text{Exp'l div} \geq 0)$

$f_*G = 0$.

$K_{X''}+B''+\lambda M'' = (f'')^*(K_X+B+\lambda M)$

$\lfloor B \rfloor = Ex(f)$.

Claim $X'' \to X$: Fano type M_i descends to X by lem 1.

$$f^*(\underbrace{K_X + B}_{\text{klt}}) = \underbrace{K_{X''} + B_0''}_{\text{sub klt}} \qquad B'' < 1$$

$\left\{ \underbrace{(X'', B'' + \lambda M'')}_{\text{lc}} \right.$ crepant model of $(X, B + \lambda M)$

$0 < \alpha \ll 1$ $\qquad \alpha B_0'' + (1-\alpha)B'' \geq 0.$

$\left\{ \begin{array}{l} (X'', \underbrace{\alpha B_0'' + (1-\alpha)B''}_{\text{klt}}) \text{ pair} \\ K_{X''} + \underbrace{\alpha B_0'' + (1-\alpha)B'' + (1-\alpha)\lambda M''}_{\text{big}} \equiv 0 / Y. \end{array} \right.$ $\Rightarrow X''$ FT / X. $\square$

W

log $\downarrow$

$(X'', B'') \rightsquigarrow K_{X''} + \Delta'' \equiv 0 / X$

$\left\{ \underline{(X'', \Delta'') \text{ klt.}} \right.$ $\square$

Proof of Thm A.

Thm A d, Φ, v — bdd family (couples)

$\exists \mathfrak{P} = \mathfrak{P}(d, \Phi, v)$ s.t. $\forall (X, B+M) \in \mathcal{G}_{glc}(d, \Phi, <v)$

$\exists$ log sm couple $(\bar{X}, \bar{\Sigma}) \in \mathfrak{P}$ & $\bar{X} \dashrightarrow X$ s.t.
① $\operatorname{Supp} \bar{\Sigma} \supseteq Ex(\bar{X} \dashrightarrow X) \cup$ st. transf of B
② M_i descends to $\bar{X}$

Step 1. $\Phi \subseteq \mathbb{R}_{>0}$ DCC.

$\exists p = p(d, \Phi, v)$, wma $pB \in \mathbb{Z}_{>0}$. pM': Cartier nef.

$$\begin{array}{ll} X' & \Gamma' = f_*^{-1}B + Ex(f) \\ f \downarrow & E + f^*(K_X + B + M) = K_{X'} + \Gamma' + M' \quad \text{for } E' \geq 0 \text{ exp'l div} \\ X & \operatorname{vol}(K_X + B + M) = \operatorname{vol}(K_{X'} + \Gamma' + M') \end{array}$$

$(X, B+M)$ replace by $(X', \Gamma' + M')$

wma. (X, B) be log sm & M_i descends to X'

Thm ([BZ16, Thm 8.1])

d. Φ DCC $\Rightarrow \exists\, \alpha = \alpha(d, \Phi)$ s.t.

- (X, B) lc of dim d
- $M' = \sum \mu_i M_i$ M_i nef Cartier & $\mu_i \in \Phi$
- $B \in \Phi$
- $K_X + B + M$ big

$\Rightarrow$ $K_X + \alpha B + \alpha M$ big.

$\exists\, \alpha = \alpha(d, \Phi)$ s.t. $K_X + \alpha B + \alpha M$ big.

fix $\beta \in (\alpha, 1)$, Φ DCC $\Rightarrow \exists\, p = p(\beta, \Phi)$ s.t.

$\forall \mu \in \Phi$, $\beta\mu < \frac{q}{p} < \mu$ for $q \in \mathbb{Z}_{>0}$.

($\inf (1-\beta)\Phi > \exists\, \sigma > 0$ let p s.t. $p\sigma > 1$.

$p(1-\beta)\mu > p\sigma > 1 \Rightarrow p\mu > p\beta\mu + 1$.

$\exists q \in \mathbb{Z}_{>0}$ s.t. $p\beta u < q < pu$

$\beta u < \frac{q}{p} < u.$

$\tau : \Phi \longrightarrow \frac{\mathbb{Z}}{p}$

$u \mapsto \frac{q}{p}$

$\tau(B) = \sum \tau(b_i) B_i$

$\tau(M') = \sum \tau(u_i') M_i'$

big $\rightsquigarrow K_X + \tau(B) + \tau(M) > K_X + \beta B + \beta M > K_X + \alpha B + \alpha M.$

& $p\tau(B) \in \mathbb{Z}$, $p\tau(M')$ Cartier

$\geq \alpha B$ $\quad \geq \alpha M$

$\forall u_0 \in (\frac{\alpha}{\beta}, 1)$ $\quad K_X + \tau(B) + u_0 \tau(M)$ big.

$(u_0 \tau(u) > \frac{\alpha}{\beta} \cdot \beta \cdot u > \alpha u)$

$(X, B + M) \rightsquigarrow (X, \tau(B) + \tau(M))$

(X, B) $\frac{1}{p}$-lc (klt). log sm.

wma $pB \in \mathbb{Z}$, pM' Cart

$K_X + B + u_0 M$ big ($u_0 < 1$)

Step 2 Find birationally bdd family.

$$K_X + 2B + 2M \text{ big } \sim_{\mathbb{Q}} \underset{\text{ample}}{\overset{\geq 0}{A}} + \overset{\geq 0}{E}.$$

$$(1+\varepsilon)(K_X + B + M) = K_X + (1-\varepsilon)(B+M) + \varepsilon(K_X + 2B + 2M)$$

$$\wr$$

$$K_X + \Delta$$

$$\sim K_X + (1-\varepsilon)(B + M) + \underline{\underline{\varepsilon A}} + \varepsilon E$$

$$= \underbrace{K_X + (1-\varepsilon)B + \varepsilon E}_{\text{klt}} + \underbrace{(1-\varepsilon)M + \varepsilon A}_{\text{ample}}$$

$$\Rightarrow \exists \Delta \sim (1-\varepsilon)B + \varepsilon E + (1-\varepsilon)M + \varepsilon A$$

s.t. (X, Δ) klt & $K_X + \Delta$ big.

BCHM (X, Δ) has minimal model (of $(X, B+M)$)

$$X \dashrightarrow X''$$

$K_{X''} + B'' + M''$ big & nef

By [BZ, Thm 1.3] $\exists\ m = m(d, p)$ s.t. $p \mid m$

$\exists\ \underbrace{L \sim m(K_{X''} + B'' + M'')}_{\mathbb{Z}\text{-div}}$ big & nef & defn a bir'l map.

$\mathrm{vol}(L) = \mathrm{vol}(m(K_{X''} + B'' + M'')) \leq m^d \cdot v$.

[Bir19, Prop 4.4]

bdd family

$\exists\ \mathcal{Q} = \mathcal{Q}(d, p, v)$ & $c = c(d, p, v) \in \mathbb{R}_{>0}$ s.t.

$\exists\ (\overline{X}, \overline{\Sigma}) \in \mathcal{Q}$ $\quad \overline{X} \overset{\bar{g}}{\leftarrow} W \overset{g''}{\rightarrow} X''$, $\overline{X} \dashrightarrow X''$ bir'l map s.t.

- $\overline{\Sigma} \supseteq \mathrm{Ex}(\overline{X} \dashrightarrow X'') \cup \mathrm{st.trsf}(B'' + L)$
- $0 < \bar{g}_* g''^* L \leq c$

$\mathcal{Q}$ bdd family, $\Rightarrow \exists\ r = r(\mathcal{Q})$ s.t. $\overline{X}$ $\bar{A}$ ample $\bar{A}^d \leq r$,

& $\boxed{\bar{A} - \overline{\Sigma}}$, $\bar{A} - \bar{g}_* g''^* L \in \overline{\mathrm{Eff}}(X)$.

Recall. $\exists\, \mu_0 < 1$ s.t. $K_{X''} + B'' + \mu_0 M''$ big

$$\frac{1}{m} L \sim_{\mathbb{Q}} K_{X''} + B'' + M'' \gtrsim \underline{(1-\mu_0) M''} \qquad \gtrsim \ \mathbb{Q}\text{-linear.}$$

($\mu_0 M'' + (1-\mu_0) M''$)

$$\bar{g}_* g''^* \frac{1}{m} L \gtrsim \bar{g}_* g''^* (1-\mu_0) M''$$

m, μ_0

$$\rightsquigarrow \quad \bar{g}_* g''^* L \gtrsim \bar{g}_* g''^* M'' \qquad \underline{m' m (1-\mu_0) > 1}$$

M' nef

$$\rightsquigarrow \quad \boxed{\bar{A} - \bar{g}_* g''^* M''} \in \overline{\mathrm{Eff}}$$

$$(X, B+M) \leftarrow (\bar{X}, \bar{B} + \bar{M}) \qquad \bar{B} = \text{st. tra } B + (1-\tfrac{1}{p})\, \mathrm{Ex}(\bar{X} \dashrightarrow X)$$

wma.

- (X, B) log sm klt
- $pB \in \mathbb{Z}_{>0}$, pM' Cartier
- v.ample A s.t. $A^d \leq r$
- $A - K_X$ & $A - (B+M) \in \overline{\mathrm{Eff}}$

($\underline{(X, B+M) \text{ not } g\text{-lc.}}$ / M' not descend to X).

Step 3. lc modification.

$\Rightarrow \exists\ c_0 = c_0(d, p, r)$ s.t.

$$(X_0, B_0 + 3dpM_0) \xrightarrow{f_0} (X, B + 3dpM) \text{ s.t.}$$

- $f_0^*M = M_0 + \sum e_i E_i \ (e_i \geq 0)$, $\sum e_i < c_0 \ (e_i > 0)$
- M' descends to X_0 & pM_0 Cartier

$\lambda = \operatorname{lct}(X, B; M)$

Step 4.

$$\begin{array}{ccc} X_0 & \dashrightarrow & Y \\ \downarrow & \swarrow_{g} & \\ X & & \end{array}$$

$(Y, B_Y + \lambda M_Y) \xrightarrow{\text{crepant}} (X, B + \lambda M)$

- $\lfloor B_Y \rfloor = \operatorname{Ex}(g)$ ← $\sum e_i < c$.
- $(Y, B_Y - t\lfloor B_Y \rfloor + \lambda M)$ klt $\forall t \ll 1$
- $(Y, B_Y) \in$ bdd family
- $A_Y - (B_Y + M_Y) \in \operatorname{Eff}(Y)$, $A_Y^d \leq r'$

$(\mathcal{X}, \mathcal{B})$ ← $\#\{x\} < +\infty$

↑

x's glet

μ not descnds $(Y, \mathcal{B}_Y)$

↑

$(Y_1, \mathcal{B}_{Y_1})$ x_1's $\#\{x_1\} < +\infty$

↑

⋮

Day 2.

Recall

Notation. $d \in \mathbb{Z}_{>0}$, $\Phi \subseteq \mathbb{R}_{>0}$ DCC, $v \in \mathbb{R}_{>0}$

$$\mathcal{G}_{glc}(d,\Phi) = \left\{ (X, B+M) \;\middle|\; \begin{array}{l} \text{g-lc } \dim X = d \\ B \in \Phi,\ M^i \in \Phi \\ K_X+B+M \text{ big} \end{array} \right\}$$

$$\mathcal{G}_{glc}(d,\Phi,<v) = \{ \ \cdots \quad \mathrm{vol}(K_X+B+M) < v \}.$$

Thm A. $\exists\, \mathcal{P} = \mathcal{P}(d,\Phi,v)$ bdd of couples s.t.

$\forall (X, B+M) \in \mathcal{G}_{glc}(d,\Phi,<v)$ $\exists$ log sm $(\overline{X}, \overline{\Sigma}) \in \mathcal{P}$, $\overline{X} \dashrightarrow X$ birat'l map

s.t.
- $\overline{\Sigma} \supseteq \mathrm{Ex}(\overline{X} \dashrightarrow X) \cup \mathrm{Supp}(\widetilde{B})$
- M_i descends to $\overline{X}$.

Idea. $(X, B+\lambda M)$ ← (X,B) klt & $pB \in \mathbb{Z}$, pM_i Cart

$\lambda_0 = \mathrm{lct}(X,B;M)$

$f_1 \uparrow$ dlt

$(X_1, \Gamma_1 + \lambda M_1) \leftarrow (X_1, B_1 + \lambda_1 M_1)$, $\Gamma_1 - \frac{1}{p}\lfloor \Gamma_1 \rfloor$, $\lambda_1 = \mathrm{lct}(X_1, B_1; M_1) > \lambda_0$

$\uparrow$ dlt

$(X_2, \Gamma_2 + \lambda_1 M_2)$

$\vdots$ $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$

terminates.

① Birt'ly bdd (X, B)

⇒ ② Construct model of (X, B) (as dlt)

③ Acc for dlt (show ② terminates)

Proof of Thm A

Step 1. $\exists\, p = p(d, \mathfrak{I}, v)$, wma. $(X, B+M)$ sat. t. f

- (X, B) log sm klt
- $pB \in \mathbb{Z}$ & pM' Cart
- $\exists$ v. ample A s.t. $A^d <^{\exists} r = r(d, p, v)$
- $A - (B+M)$ p eff.

Step 2. ②: Find a bdd family (dlt mod)

- $\lambda = \mathrm{lct}(X, B; M) < +\infty$ (M' not descends to X)

$\Rightarrow \exists\, s = s(d, p, r)$ s.t. f

$\exists\ (Y, B_Y + \lambda M_Y) \xrightarrow[g]{\text{one part}} (X, B + \lambda M)$

- $\mathrm{Ex}(g) = \lfloor B_Y \rfloor$
- $(Y, B_Y - t\lfloor B_Y \rfloor + \lambda M_Y)$ klt $\forall\, 0 < t \ll 0$ ($t = \frac{1}{p}$ $\underline{B_Y - t\lfloor B_Y \rfloor}$).

Claim 2

$\exists\, \alpha_0 = \alpha_0(d, p, r)$

s.t. $(X, B + \alpha_0 M)$ klt.

apply

$\Rightarrow \exists\, \alpha_0'$

$(Y, \alpha_0' M_Y)$ klt

• $\exists$ a v.ample A_Y w/ $A_Y^d \leq s$ & $A_Y - (B_Y + M_Y)$ pseff

Step 3. Finish the proof of Thm A.

Claim 1. $\# \{ \lambda = \text{lct}(X, B; M) \} < +\infty$.

Proof. $g : (Y, B_Y + \lambda M_Y) \to (X, B + \lambda M)$ crepant.

Fact 1) g not isom. (if g is iso, $(Y, B_Y - \lfloor B_Y \rfloor + \lambda M_Y) = (Y, B_Y + \lambda M_Y)$ klt; $\Downarrow$ crepant; $(X, B + \lambda M)$ lc non-klt $\Rightarrow \Leftarrow$)

2) $K_Y + B_Y$ not nef.

If nef. neg. lem $g^*(K_X + B) = K_Y + B_Y + (\geq 0)$

$(g^*(K_X + B) - (K_Y + B_Y))^{\geq 0}$ anti-nef $/X$ & exc'l $/X$)

(X, B) klt (Y, B_Y) lc not klt. $\Rightarrow \Leftarrow$

$\lfloor B_Y \rfloor \neq 0$.

$\exists$ curve $C \to pt$, $0 > (K_Y + B_Y) \cdot C \geq -2d$

$(Y, B_Y) \in$ Bdd family, Cartier ind of $K_Y + B_Y$ is bdd

$\Rightarrow \# \{ -(K_Y + B_Y) \cdot C > 0 \} < +\infty$.

$F_Y + B_Y + \lambda M_Y \equiv 0 / X$ ← C finite set

$(\qquad) \cdot C = 0 \Rightarrow \lambda = \frac{-(F_Y + B_Y) \cdot C}{M_Y \cdot C} < 3dp$

1, s.t. Cart ind of M_Y is bdd.

Claim 2 $\exists \alpha_0 = \alpha_0(d, p, s)$ s.t. $(Y, \alpha_0 M_Y)$ klt.

($\exists \alpha_0 = \alpha_0(d, p, r)$ s.t. $(X, B + \alpha_0 M)$ klt. More gen'l)

Assume Claim 2 ($\alpha_0 \in \mathbb{Q}$)

$(Y, \alpha_0 M_Y)$ klt $\quad F_Y + \alpha_0 M_Y + 2d A_Y$ nef. (length of extr'l ray)

$(F_Y + \alpha_0 M_Y + 2d A_Y) A_Y^{d-1} \le \exists$ a bound (A_Y - F_Y pseff, A_Y - M_Y pseff)

$\Rightarrow$ $F_Y + \alpha_0 M_Y + 2d A_Y$ has bdd Cart ind

$\Rightarrow$ M_Y has bdd Cart ind.

Proof Claim 2 (Lem 2.25 L ref)

$M_Y = M_Y^+ - M_Y^-$

$= (F_Y + \alpha_0 M_Y + 2d A_Y) - (\quad)$

[Thm 8 Birkar]

$d, r \in \mathbb{Z}_{>0}$, $\varepsilon > 0$. $\exists\, t = t(d, r, \varepsilon)$ s.t. if

- (X, B) ε-lc of dim d
- A v.ample s.t. $A^d \leq r$
- $A - B$ p-eff
- $M \geq 0$ $\mathbb{R}$-Cart $\mathbb{R}$-div s.t. $A - M$ p-eff.

$\Rightarrow$ $\mathrm{lct}(X, B, |M|_{\mathbb{R}}) \geq t$ $\left((X, B + \frac{t}{2} M) \text{ klt} \right)$

<u>Continue.</u>

$$X' \xrightarrow{f} X$$

$$\left\{ \begin{aligned} & f^*(K_X + B) = K_{X'} + B' \\ & f^* M = M' + E' \\ & f^* A \sim_{\mathbb{Q}} \boxed{A'_k} + \frac{1}{k} \underline{\underline{G'}}_{\geq 0} \quad (A'_k \text{ ample}) \end{aligned} \right.$$

sub klt

$$f^*(\underbrace{K_X + B + \lambda_0 M}_{\text{klt}}) = \left(K_{X'} + B' + \lambda_0 E' + \underline{\underline{\lambda_0 M'}} + \underline{\underline{\frac{\lambda_0}{k} G'}} + \underline{\underline{\lambda_0 A'_k}} \right)$$

Since $M' + A'_k$ ample $\Rightarrow$ $\exists\, H' \underset{\mathbb{Q}}{\sim} M' + A'_k$ s.t. $\underline{\underline{(X', B' + \lambda_0(E' + \frac{1}{k} G' + H'))}}$ sub-klt

Note. $H = f_* H'$, $G = f_* G'$

$$H + \frac{G}{k} = f_*\left(H' + \frac{G'}{k} + E'\right) \ \& \ H' + \frac{G'}{k} + E' \sim_{\mathbb{Q}} f^*(A+M) \equiv 0/X$$

(above: $\sim_{\mathbb{Q}} M' + A_k'$)

$$\Rightarrow \quad f^*\left(H + \frac{G}{k}\right) = H' + \frac{G'}{k} + E'$$

$$f_*\left(K_{X'} + B' + \lambda_0\left(E' + \frac{1}{k}G' + H'\right)\right) = K_X + B + \lambda_0\left(H + \frac{G}{k}\right)$$

$$K_{X'} + B' + \lambda_0\left(E' + \frac{1}{k}G' + H'\right) = f^*\left(K_X + B + \lambda_0\left(H + \frac{G}{k}\right)\right)$$

sub klt $\qquad\qquad$ klt

$$2A - \left(H + \frac{G}{k}\right) \sim_{\mathbb{R}} 2A - (A+M) = A - M \ \text{peff}.$$

$\left(H + \frac{G}{k}\right) \geq 0$

(X, B)

[Thm 8 Bir21]

$d, r \in \mathbb{Z}_{>0}$, $\varepsilon > 0$. $\exists t = t(d, r, \varepsilon)$ s.t. if

- (X, B) ε-lc of dim d
- A v.ample s.t. $A^d \leq r$
- $A - B$ peff
- $M \geq 0$ $\mathbb{R}$-Cart $\mathbb{R}$-div s.t. $A - M$ p-eff.

lct $(X, B, |M|_{\mathbb{R}}) \geq t$ $\quad \left((X, B + \frac{t}{2}M) \text{ klt} \right)$

$\Rightarrow \exists \alpha_0 = \alpha_0(d, p, r)$ s.t. $\left(X, B + \alpha_0\left(H + \frac{G}{k}\right)\right)$ klt

$\Rightarrow \quad \underline{\lambda_0 \geq \alpha_0}$ $\qquad \square$

$$\lambda = \frac{(-(K_X+B)\cdot C)\cdot R}{(M_Y\cdot C)\cdot R}$$ ← forget

$\#\{-(K_X+B)\cdot C\}$ finite set.

3dp $\lambda = \frac{m}{n} > \alpha_0$.

Claim 1 $\#\{\lambda = \mathrm{lct}(X, B; M)\} < +\infty$

$\exists (X_i, B_i + M_i)_{i\in I}$ that M_i not descends.

$\#\{\lambda_i = \mathrm{lct}(X_i, B_i; M_i)\}_{i\in I} < +\infty$.

$\Rightarrow \exists I_1 \subseteq I$ s.t.

$\lambda_j = \lambda_{i_0}$ (for some $i_0 \in I$) $\forall j \in I_1$

Step 2. $(X_i, B_i + \lambda_i M_i)$ $(i \in I_1)$

↑

$(Y_i', \Gamma_{Y_i'} + \lambda_i M_{Y_i'})$ & $(Y_i', B_{Y_i'} + \lambda_i M_{Y_i'})$ klt g-pair

$B_{Y_i'} = \Gamma_{Y_i'} - \frac{1}{p}\lfloor \Gamma_{Y_i'} \rfloor$

$\lambda_i' := \mathrm{lct}(Y_i', B_{Y_i'}; M_{Y_i'}) > \lambda_i = \lambda_{i_0}$

Step 2 $\#\{\lambda_i'\} < +\infty \Rightarrow I_2 \subseteq I_1$ s.t.

$\lambda_k' = \lambda_{i_1}'$ (for some $i_1 \in I_1$) $\forall k \in I_2$

$(y_i^1, B y_i^1 + \lambda_i^1 M y_i^1)$ $(i \in I_2)$

$\uparrow$ crep

$\Gamma_{y_i^1} - \frac{1}{p} L \Gamma_{y_i^2} J$

$\shortparallel$

$(y_i^2, \Gamma_{y_i^2} + \lambda_i^1 M y_i^2)$ & $(y_i^2, B y_i^2 + \lambda_i^1 M y_i^2)$ krit

$\lambda_i^2 = \mathrm{lvt}(y_i^2, B y_i^2 ; M y_i^2) > \lambda_i^1 = \underline{\lambda_{i_1}^1} > \lambda_{i_0}$

$\# \{ \lambda_i^2 \}_{i \in I_2} < +\infty$

$\Rightarrow I_3 \subseteq I_2$ s.t. $\lambda_k^2 = \lambda_{i_2}^2$ (for some $i_2 \in I_2$) $\forall k \in I_3$.

$\uparrow$

$(y_i^3, \Gamma_{y_i^3} + \lambda_i^2 M y_i^3)$ ---

$\cdots > \cdots > \lambda_{i_3}^3 > \lambda_{i_2}^2 > \lambda_{i_1}^1 > \lambda_{i_0}$

$B \in \frac{2}{p}$ & pri Conti $\Rightarrow$ Acc for g-lvt $\Rightarrow \Leftarrow$

step 2. apply $(Y_j, B_{Y_j} + \lambda'_j M_{Y_j})_{j \in I_2}$

$\uparrow$ $\qquad T_{Y_j^2} - \frac{1}{p}[\cdot]$

$(Y_j^2, T_{Y_j^2} + \lambda'_j M_{Y_j^2})$ & $(Y_j^2, B''_{Y_j^2} + \lambda'_j M_{Y_j^2})$ klt

(Claim 1) & $\#\{\lambda_j^2 = \mathrm{lct}(Y_j^2, B_{Y_j^2}; M_{Y_j^2})\} < +\infty$

$I_3 \subseteq I_2$, $\underbrace{\lambda_j^2 = \lambda^R_{i_2} > \lambda'_{i_1} > \lambda_{i_0}}_{\forall j \in I_3}$

Claim: $(X, B+\lambda M)$ sati (✪)

$\uparrow$ step 2

$(Y', B_{Y'} + \lambda M_{Y'})$ satify (✪)

$\uparrow$ step 2

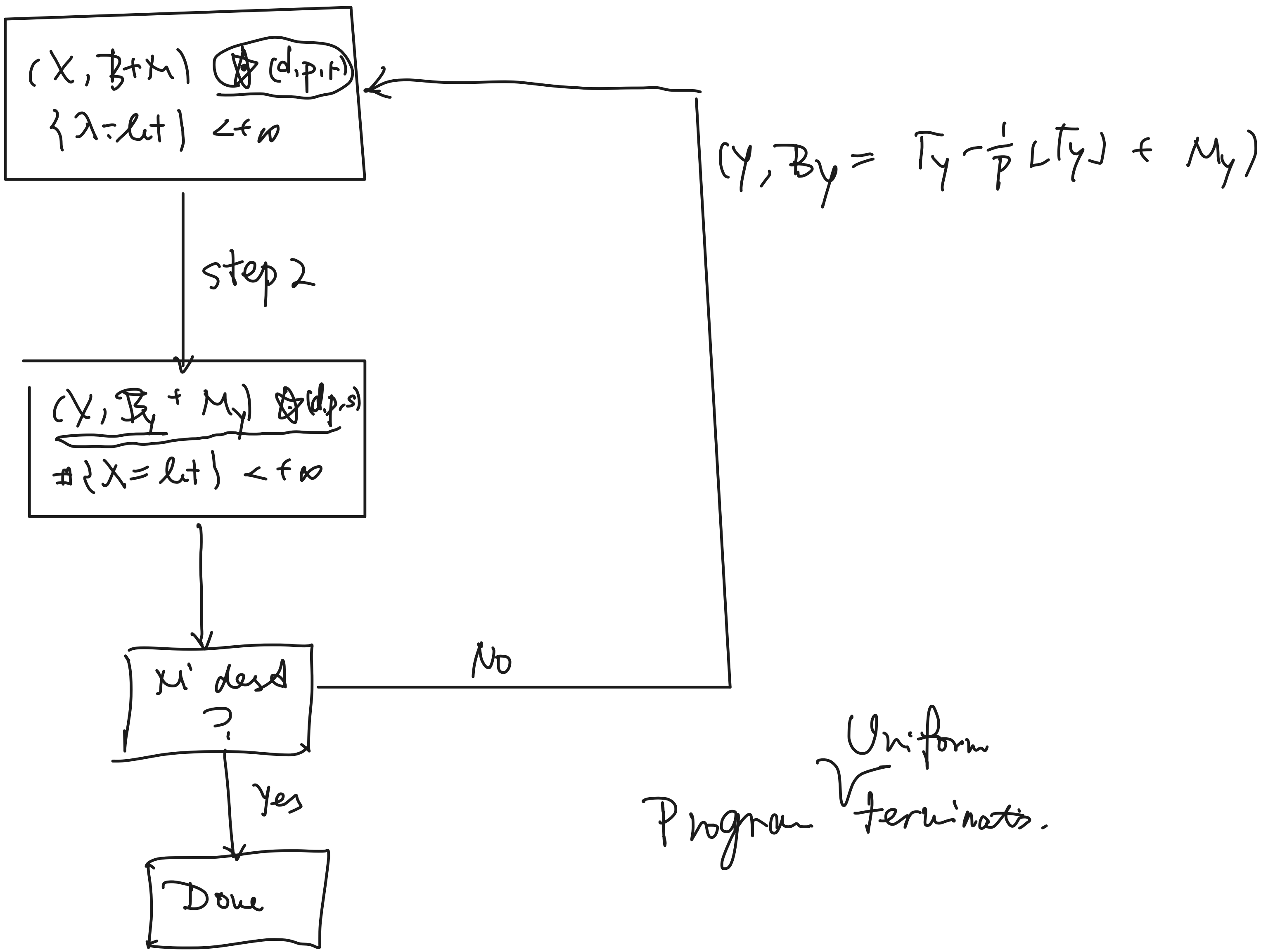
$(Y^2, B_{Y^2} + \lambda' M_{Y^2})$ satify (✪)

$\uparrow$

$\vdots$

$pB \in \mathbb{Z}$

pM Cartier



Program uniform terminates.

If no terminates $\{(X_i, B_i + M_i)\}_{i \in I}$ $\quad \lambda_{i_0} < \lambda^1_{i_1} < \lambda^2_{i_2} < \lambda^3_{i_3} < \cdots$

lct Acc.

Step 2. ② : Find a bdd family (klt mod)

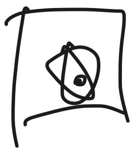
- $\lambda = \mathrm{lct}(X, B; M) < +\infty$ (M' not descends to X)

$\Rightarrow \exists\, s = s(d, p, r)$ s.t.

$$\exists\ (Y, B_Y + \lambda M_Y) \xrightarrow[g]{\text{onepoint}} (X, B + \lambda M)$$

- $E_X(g) = \lfloor B_Y \rfloor$
- $(Y, B_Y - t\lfloor B_Y \rfloor + \lambda M_Y)$ klt $\forall\, 0 < t \ll 0$ ($t = \frac{1}{p}$ $\quad \underline{B_Y^- + \lfloor B_Y \rfloor}$)

Prop 7.5



- (X, B) klt
- $pB \in \mathbb{Z}$ & pM' Cart
- $\exists$ v. ample A s.t. $A^d < \exists\, r = r(d, p, v)$
- $A - (B + M)$ p eff.

$\Rightarrow \exists\, C_0 = C_0(d, p, r)$ s.t. $X'' \xrightarrow{f''} X$

g-lc $(X'', B'' + 3dpM'')$, s.t. (• $\lfloor B'' \rfloor = E_X(f'')$ • $K_{X''} + B'' + 3dpM''$ ample $/X$) lc model

not bdd

- $f''^*M = M'' + \underline{\sum e_i E_i}$, $0 < e_i$, $\sum e_i < C_0$
- M' descends to X'' & pM'' Cart

($X'' \dashrightarrow Y$ (in step 2). $X'' \to X$, $Y \to X$)

$\mathcal{Y}$

$f \downarrow$

$\mathcal{X}$

μ' closed $\bar{X}$

$\exists$ $\underline{M}$ nef on $\mathcal{X}$ — $\bar{M} \overset{?}{=} M|_{\bar{X}}$ ✓

$f^*\bar{M} = M_Y + \sum e_i E_i \quad \sum e_i \leq \exists ?$

Thm A $\Longrightarrow$ Prop 3.5 $3dp \longleftrightarrow$ some number ?

$+ \ (X'', B'')$ bdd

Proof of Prop 3.5 g.e.

$K_{X'}+\Delta'+H'+3dpM$ – MMP $= K_{X'}+\Delta'$ MMP

$\mu' \curvearrowright X' \dashrightarrow X''$ ($3dpM+H'$) trivial MMP/X

$f \downarrow$

(X,B) klt $\quad X$

$K_{X'}+\Delta'$ – MMP.

$\Delta' = \tilde{B} + E_X(f)$

$H' \sim_{\mathbb{Q}} f^* 3dA$

$(2H' \in |f^* 3dA|)$

Day 3

Step 1.

- (X, B) log sm klt ($\mathbb{Q}$-fact)
- $pB \in \mathbb{Z}$ & pM Cartier
- $\exists A$ w/ $A^d \leq \exists r = r(d, \Phi, v)$ & $A - (M+B)$ pseff.

Step 2 $\lambda = \mathrm{lct}(X, B; M) < +\infty$

$\exists (Y, B_Y + \lambda M_Y) \xrightarrow{h} (X, B + \lambda M)$ s.t.

- $\lfloor B_Y \rfloor = \mathrm{Ex}(h)$
- $(Y, B_Y - t\lfloor B_Y \rfloor + \lambda M_Y)$ klt $\forall 0 < t \ll 1$

$\Rightarrow$ • $A_Y^d \leq \exists s = s(d, p, r)$ & $A_Y - (B_Y + M_Y)$ p-eff.

① $\exists$ lc model $(X_0, B_0 + 3dpM_0) \xrightarrow{f_0} (X, B + 3dpM)$ ✓ • $\lfloor B_0 \rfloor = \mathrm{Ex}(f_0)$ • $K_{X_0} + B_0 + 3dpM_0$ ample$/X$ (lc)

Prop 3.5

- $f_0^* M = M_0 + \sum e_i E_i$, $\sum e_i \leq \exists c = c(d, p, r)$

✓ M' descends to X_0. (2 places)

Proof of ①

$(X', B' + 3dpM') \overset{K_{X'} + B' + 3dpM'}{\dashrightarrow} X''$ (E_i-contr), $f_*^{-1}B + \mathrm{Ex}(f)$, $(f_*E_i'') = 0$

$f \downarrow$, $X' \to X$; $f'': X'' \to X$

- M' descends at each step
- $f''^*(K_X + B + 3dpM) = K_{X''} + B'' + 3dpM'' + G''$ (≥ 0)

$$K_{X'} + B' + 3dpM' = f^*(K_X + B + \overset{3dp}{\cancel{\,}}M) + \underset{=}{\overset{\geq 0}{E_1}} - \underset{=}{\overset{\geq 0}{E_2}}$$

$$\equiv E_1 - E_2 \,/X$$

<u>Claim</u>. $f''^* M = M'' + \sum e_i E_i$, $e_i > 0\ \forall i$ (Neg $e_i \geq 0$)

(Otherwise, $e_1 = 0 = \mathrm{mult}_{E_1}(f''^* M - M'')$ $\qquad \underline{\lfloor B' \rfloor = \mathrm{Ex}(f)}$

$$0 < a(E_1, \underline{X, B}) = a(E, X, B) - \overset{3dp}{\,}\mathrm{mult}_E(f''^* M - M'')$$

$$= a(E, X, B + 3dpM)$$

$$= 1 - \mathrm{mult}_{E_1}(\underline{B''} + C'') \leq 0. \Rightarrow\Leftarrow \text{)}.$$

Prot $0 < t \ll 1$,

$$\underline{K_{X'} + B' - \underline{t\sum e_i E_i} + (3dp - t)M'} \text{ dlt}$$

$$\|$$

$$K_{X'} + B' + 3dpM' - tf^* M. \equiv K_{X'} + B' + 3dpM' \,/X.$$

MMP

$X' \dashrightarrow X''$

$\downarrow$

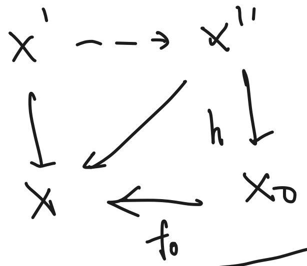
X

$\lfloor B'' - t\sum e_i E_i \rfloor = \phi.$

$(X'', B'' - t\sum e_i E_i + (3dp - t)M'')$ klt

$\}$ big $/X$

$\Rightarrow$ MMP ter w/ a gmm.



wma X'' is the gmm $K_{X''}+B''-t\sum e_iE_i+(3dp-t)M''$ ample model of $K_{X''}+B''+3dpM''$

$h^*(K_{X_0}+B_0+3dpM_0) = K_{X''}+B''+3dpM''$.

X'' FT $/X_0$

M' des to X_0

$\Leftarrow$ { $h^*(\underbrace{K_{X_0}+B_0+(3dp-t)M_0 - t\sum e_iE_i}_{klt}) = \underbrace{K_{X''}+B''-t\sum e_iE_i+(3dp-t)M''}_{klt}$

$-(\underbrace{K_{X''}+B''-t\sum e_iE_i}_{klt}) \equiv (3dp-t)M''$ big & nef $/X_0$

$K_{X_0}+B_0+3dpM_0$ ample $/X$

Claim: $K_{X_0}+B_0+A_0+3dpM_0$ ample (globally) (lc)

$A_0 \sim 3d f_0^*A$

If not, $\Rightarrow \exists C$ s.t. $(K_{X_0}+B_0+A_0+3dpM_0)\cdot C \le 0$.

$\Rightarrow f_*C \ne pt$.

Negative Ext'l ray of $(K_{X_0}+B_0+3dpM_0)$ R $\quad f_*R \ne pt$ $\quad \Rightarrow\Leftarrow$.

$$\left\{\begin{array}{l} C_0\cdot(K_{X_0}+B_0+3dpM_0) > -2d \\ C_0\cdot A_0 = f_{0*}C_0\cdot 3dA > 3d \end{array}\right\} \Rightarrow (K_{X_0}+B_0+A_0+3dpM_0)\cdot R > 0.$$

$K_{X_0} + B_0 + A_0 + 3d_p M_0$ ample (global)

$\forall E \subseteq \lfloor B_0 \rfloor$ (E^{ν})

$$(K_{X_0} + B_0 + A_0 + 3d_p M_0)|_E = (K_E + B_E + A_E) + (3d_p M_0)|_E \text{ ample.}$$

lc ($K_{X_0}+B_0+A_0$), $p(\cdot) \in \mathbb{Z}$; $K_E + B_E + A_E$ ample, $\in \Phi_0$ DCC set.

$\Rightarrow \exists\, \alpha_0 = \alpha_0(d-1, \Phi_0, p)$ s.t. $\mathrm{vol}(\ \ ) \geq \alpha_0$ ← BZ ✓

[Birkar–Zhang]

$$\mathrm{vol}(\ \ ) = (K_{X_0} + B_0 + A_0 + 3d_p M_0)^{d-1} \cdot E$$

(if $X_0 \xrightarrow{f_0} X$ small,

$$f_0^*(K_X + B + 3d_p M) = K_{X_0} + B_0 + 3d_p M_0. \quad f_0 = \text{iso}.$$

(ample / X)

$f_0^* M = M_0 \Rightarrow$ M' desc. to X $\Big\} \Rightarrow \Sigma e_i = 0.$

$(e_i > 0 \ \ \#\{E_i\} \neq \phi)$)

$$(\Sigma e_i) \cdot \alpha_0 = \Sigma(e_i \cdot \alpha_0) \leq (\Sigma e_i E_i)(K_{X_0} + B_0 + A_0 + 3d_p M_0)^{d-1}$$

$$\leq (M_0 + \Sigma e_i E_i)(K_{X_0} + B_0 + A_0 + 3d_p M_0)^{d-1}$$

$$\overset{\parallel}{f_0^* M \cdot (\ \cdots\ )^{d-1}}$$

A-M pdt.

$$\Rightarrow \leq f_0^* A \cdot (K_{X_0} + B_0 + A_0 + 3d_p M_0)^{d-1}$$

(← nef; ample)

$$\leq (f_0^* A + F_{X_0} + B_0 + A_0 + 3dp M_0)^d$$

$$= \mathrm{vol}(f_0^* A + F_{X_0} + B_0 + A_0 + 3dp M_0)$$

$$\leq \mathrm{vol}(f_0^* A + f_0^*(F_X + B + 3d A + 3dp M))$$

$$= \mathrm{vol}(A + F_X + B + 3d A + 3dp M)$$

$x \in B_{\mathrm{dd}}$

$A - F_X$ psef

$$\leq \mathrm{vol}(A + A + A + 3d A + 3dp A)$$

$$\leq (3 + 3d + 3dp)^d \cdot r$$

$$c = \frac{(3 + 3d + 3dp)^d \cdot r}{\alpha_0}$$

$$\Rightarrow \sum e_i \leq c.$$ ☐

In Birkar's pape.

$X' \dashrightarrow X'' \to X_0$

$F_{X'} + B' + H' + 3dp M' - \mathrm{MMP} = (\ )$ - MMP / X

$f \downarrow$

X

$B' = f_*^{-1} B + \mathrm{Ex}(f)$

$H' \sim_{\mathbb{Q}} 3d f^* A$ s.t. $F_{X'} + B' + H'$ lc

$H' \sim 6d f^* A$ & $H' \in |\frac{1}{2} H'|_{\mathbb{Q}}$

Cont to step 2.

<u>Step 2</u> $\lambda = \mathrm{lct}(X, B; M) < +\infty$

$\exists\, (Y, B_Y + \lambda M_Y) \xrightarrow{h} (X, B + \lambda M)$ s.t.

- $\lfloor B_Y \rfloor = Ex(h)$ ✓
- $(Y, B_Y - t\lfloor B_Y \rfloor + \lambda M_Y)$ klt $0 < t \ll 1$

$\Rightarrow$ • $A_Y^d \leq \exists\, s = s(d, p, r)$ & $A_Y - (B_Y + M_Y)$ p-ff.

① $\Rightarrow$ $(X_0, B_0 + 3dp M_0) \dashrightarrow X_{l-1} \dashrightarrow X_l = X$

$K_{X_0} + B_0$ MMP w/ scaling of M_0

X_0: $\mathbb{Q}$-fact, lc mod $f_0 \downarrow X$

$B_0 = f_{0*}^{-1} B + Ex(f_0)$ & (X, B) klt

$$\begin{array}{ccc} X' & \dashrightarrow & X'' \\ \downarrow & & \downarrow h \\ X & \leftarrow & X_0 \end{array}$$

$\underline{K_{X''} + B'' - t\sum e_i E_i + (3dp - t) M''} = h^*(\underline{K_{X_0} + B_0 - t\sum e_i E_i + (3dp - t) M_0})$

klt ... klt.

(X, B) klt $\underline{K_{X_0} + B_0 - f_0^*(K_X + B) \geq 0}$ & exc'l & Supp $= Ex(f)$

$$\begin{array}{ccc} X_0 & \overset{Ex(f_0)\text{-cont}}{\dashrightarrow} & X_0' \\ f_0 \downarrow & \swarrow & \\ X & & \end{array} \qquad \begin{array}{c} X_0' \\ f_0' \downarrow \\ X_0 \end{array}$$

Since X_0: $\mathbb{Q}$-fact

$\Rightarrow \exists\, H$ ample $/X_0$ & Exp $/X_0$.

if f_0' not iso, $\exists\, C \to pt$ $0 < C \cdot H$

$$\begin{array}{ccccccc} & \alpha_1 & & \cdots & \alpha_{l-1} & & \\ X_0 & \dashrightarrow X_1 \dashrightarrow & & \dashrightarrow & X_{l-1} & \xrightarrow[E_l]{h_l} & X_l \end{array}$$

h_l ← ext'l contraction.

$X_0 \to \hat{X}$, $X_l \to \hat{X}$ ($=$)

$K_{X_0}+B_0$ w.s of M_0

s.t. $K_{X_{l-1}}+B_{l-1}+\alpha_{l-1}\cdot M_{l-1} \equiv 0 \;/ X_l = X$.

$\Downarrow$ (X,B) klt

$h_l^*(K_{X_l}+B_l+\alpha_{l-1}M) = K_{X_{l-1}}+B_{l-1}+\alpha_{l-1}M_{l-1}$

g-lc non-klt; $\lfloor B_{l-1}\rfloor \neq 0$. lc

$\Rightarrow \alpha_{l-1} = \lambda = \mathrm{lct}(X, B; M)$

take i min'l s.t. $\alpha_i = \lambda < \alpha_{i-1}$

$X_0 \dashrightarrow (X_i, B_i+\lambda M_i) = (Y, B_Y+\lambda M_Y)$ $\quad (K_{X_0}+B_0+\lambda M_0)$-MMP

$\downarrow$

X

$\lambda = \alpha_i < \alpha_{i-1} \le \alpha_{\cdots} \le \alpha_1$ ← i min'l

- $X_0 \dashrightarrow X_i$ MMP on $K_{X_0}+B_0$ w.s of M_0 ($+\lfloor B_0\rfloor$)

$(X_0, B_0 - t\sum e_iE_i + (3dp-t)M_0)$ klt & M' des to X_0

Lc ct $(X_0, B_0+(3dp-t)M_0)$ = Lc center of $(X_0, B_0) = \mathrm{Supp}\lfloor B_0\rfloor$

$\|$

$\Rightarrow$ Lc cent of $(X_i, B_i+\lambda M_i) = (Y, B_Y+\lambda M_Y) = \lfloor B_Y\rfloor$.

$\Rightarrow (Y, B_Y - t\lfloor B_Y\rfloor + \lambda M_Y)$ klt. $\forall\, 0<t\ll 1$.

$$\text{I, s.t. } \exists s^{s(d.p.r)} \text{ s.t. } \begin{cases} A_Y^d \le s \\ A_Y - (B_Y + M_Y) \text{ psef.} \end{cases}$$

$$\left\{ \begin{array}{l} \sum e_i \le c. \\ \lambda = \mathrm{lct}(X; B; M) \ge^{\exists} \lambda_0 = \lambda_0(d.p.r). \end{array} \right\} \Rightarrow \exists \mu < \lambda_0$$

$$K_Y + B_Y + \lambda M_Y = f_Y^*(K_X + B + \lambda M)$$

$$K_Y + B_Y - \mu \sum e_i E_i + (\lambda - \mu) M_Y = f_Y^*(K_X + B + (\lambda - \mu) M_Y).$$

≥ 0 $\qquad \frac{\mu \varepsilon}{\lambda}$-lc

$$K_X + B + (\lambda - \mu) M$$

$$= \frac{\mu}{\lambda}(K_X + B) + \frac{\lambda - \mu}{\lambda}(K_X + B + \lambda M)$$

$$\Rightarrow (X, B + (\lambda - \mu) M) \ \frac{\mu}{\lambda}\varepsilon\text{-lc.}$$

$\begin{cases} pB \in \mathbb{Z} \\ (X, B) \in \text{log bdd klt} \end{cases}$

$\Rightarrow (X, B)$ ε-lc

$\varepsilon = \varepsilon(d, r, p)$

[Bir'18 Thm 2.2]

Def (d, r, ε)-FT fib $(X, B + M) \xrightarrow{f} Z$ s.t. $\begin{cases} (X, B + M) \ \varepsilon\text{-lc} \\ K_X + B + M \sim_{\mathbb{R}} f^* L \\ X/Z \text{ FT} \end{cases}$

$\exists A$ on Z s.t. $A^{\dim Z} \le r$ &

$\cdot A - L$ ample.

$\tau > 0$.

Thm 2.2. (d, r, ε)-FT fib $(X, B+M) \to Z$ s.t.

- $0 \le \exists \Delta \le B$ & $\Delta \ge \tau$
- $-(K_X + \Delta)$ big$/Z$

$\Rightarrow (X, \Delta)$ log bdd.

$\Rightarrow (Y, B_Y) \in$ log bdd.

$\exists s = s(d, p, r)$ s.t. $\exists A_Y$ w/ $A_Y^d \le s$. & $A_Y - (B_Y + M_Y)$ pseff. ☑

$A - K_X$ pseff.

$A - (B+M)$ pseff.

$A_Y - f_Y^* A$ pseff

$$K_Y + B_Y + \lambda M_Y = f^*(\underline{K_X + B + \lambda M})$$

$\lambda < 3dp$.

A

$\lambda < d_{i-1}$

$$\underline{K_{X_j} + B_j - + \sum e_i E_i + (3dp - t) M_j}$$

"

$$K_{X_j} + B_j + 3dp M_j \,/ X.$$

$X \dashrightarrow X'$

$K_{X_i} + B_i + \lambda M_i$ lc.

$K_{X_i} + B_i + \sum e_i E_i + \underline{(\lambda - t)} M_i$ klt ($(3dp - t) M_i$)

$\underline{K_{X_i} + B_i} - + \sum e_i E_i + \lambda M_i$ klt

$-+ \sum e_i E_i$ klt

$X_0 \dashrightarrow X_i$ $\underline{K_{X_0} + B_0 + \lambda M_0}$ - MMP

$X_0 \dashrightarrow X_i = Y$ $\quad$ $\mathrm{Ct}(Y, B_Y + x M_Y) \overset{\leq}{=} \mathrm{Supp}\lfloor B_Y \rfloor$

$K_{X_0} + B_0$ MMP w.r.t. of M_0 $\quad$ $\mathrm{Lc}(X, B_0 + 3dp M_Y) = \mathrm{Supp}\lfloor B_0 \rfloor$

$(X_0, \underline{B_0 + \sum e_i E_i} + \underline{(\ )})$ klt

des

$X_0 \dashrightarrow X_i$ $\quad$ $K_{X_0} + B_0 + x M_0$.

Day 4 d-Φ.

Thm B $\{ \mathrm{vol}(K_X + B + M) \mid (X, B+M) \in \mathcal{G}_{glc}(d, \Phi) \}$ satisfies DCC.

Prop 4.2 (P31).

Step 1 Reduce to Prop 4.2.

$\exists (X_i, B_i + M_i)$ w/ $v_i \downarrow$ st. $M_i = \sum_j \mu_{ij} M_{ij}$ & $M_{ij} \not\equiv 0$

Thm A $\downarrow f_i$

$v_0 > v_i$: $\exists$ $(\bar{X}_i, \bar{\Sigma}_i)$, $\bar{M}_{ij}$ des & $\bar{A} - \bar{M}_{ij}$ psef.

$(\bar{X}_i, \bar{\Sigma}_i) \in \mathcal{X}$ Claim $\exists \mathcal{M}_j$ on $\mathcal{X}$ st. $\mathcal{M}_j |_{\bar{X}_i} \sim_{\mathbb{Q}} \bar{M}_{ij}$ $\forall i, j$

$\downarrow$

T

$K_{\bar{X}_i} + \bar{M}_{ij} + 3d\bar{A}$ & $K_{\bar{X}_i} + 2\bar{M}_{ij} + 3d\bar{A}$ ample

Eff. bpf $\exists n = n(d)$ st $|n(K_{\bar{X}_i} + 2\bar{M}_{ij} + 3d\bar{A})|$ & $|n(K_{\bar{X}_i} + \bar{M}_{ij} + 3d\bar{A})|$ bpf

$n\bar{M}_{ij} \sim E_{ij} - F_{ij} \not\equiv 0$.

$\deg_{\bar{A}}(E_{ij} + F_{ij}) < \exists$ bdd $\quad (\bar{A}^{d-1} \cdot E_{ij} < \exists$ bdd$)$

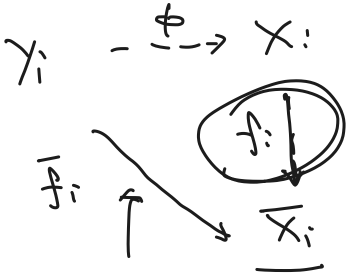
$\bar{A}^{d-1} \cdot (E_{ij} - F_{ij}) > \exists$ bdd > 0 $= \frac{1}{n} \bar{A}^{d-1} \cdot M_{ij}$

$\exists\ \mathcal{E}_j$ & $\mathcal{F}_j$ s.t. $\mathcal{E}_j|_{\bar{X}_i} = E_{ij}$ & $\mathcal{F}_j|_{\bar{X}_i} = F_{ij}$.

$\rightsquigarrow$ Set $\mathcal{M}_j = \frac{1}{n}(\mathcal{E}_j - \mathcal{F}_j) \rightsquigarrow \mathcal{M}_j|_{\bar{X}_i} \sim_a \bar{M}_{ij} \quad \forall i, j.$

standard argument $\rightsquigarrow$ $(\bar{\mathcal{X}}, \bar{\Sigma})$ log sm $\downarrow$ T

strata of $(\bar{\mathcal{X}}, \bar{\Sigma}) \overset{1:1}{\longleftrightarrow} (\bar{X}_i, \bar{\Sigma}_i)$ strata



seq of sm bl up

ext D

$$f_i^*(K_{\bar{X}_i} + \bar{B}_i + \bar{M}_i) = K_{X_i} + B_i + M_i + G_i, \quad G_i \geq 0$$

$\forall D \subseteq \operatorname{Supp} G_i$

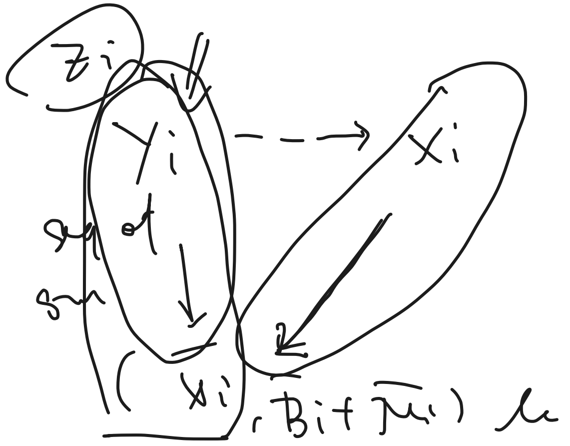
$\rightsquigarrow a(D, \bar{X}_i, \bar{\Sigma}_i) \leq a(D, \bar{X}_i, \bar{B}_i + \bar{M}_i)$

$= a(D, B_i + M_i + G_i) < 1.$

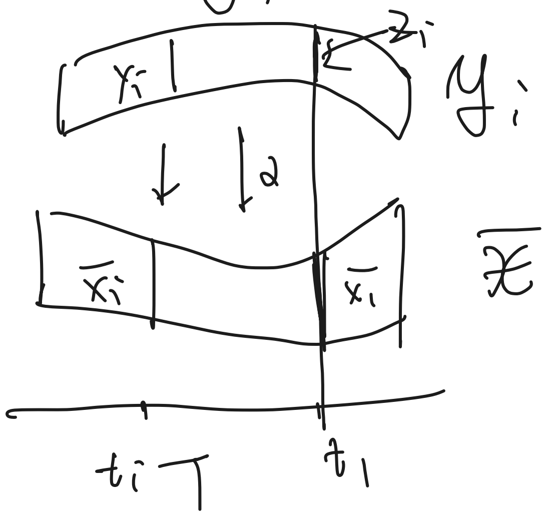
$\rightsquigarrow a(D, \bar{X}_i, \bar{\Sigma}_i) = 0$ & $\operatorname{Cent}_{\bar{X}_i} D =$ stratum of $(\bar{X}_i, \bar{\Sigma}_i)$

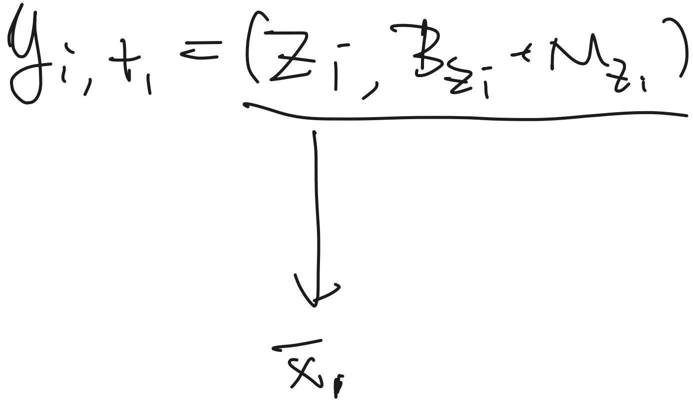
$B_{Y_i} = \phi_*^{-1} B_i + \operatorname{Ex}(\phi) \qquad M_{Y_i} = \bar{f}_i^* \sum \mu_j \bar{M}_{ij}$

Claim: $\mathrm{vol}(K_{Y_i}+B_{Y_i}+M_{Y_i}) = \mathrm{vol}(K_{X_i}+B_i+M_i)$

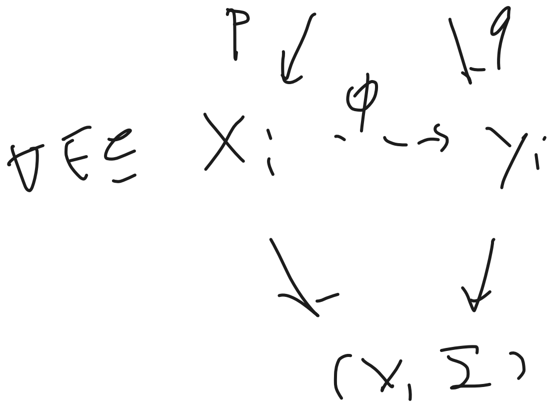


$\rightsquigarrow \quad \mathcal{Y}_i \longrightarrow \overline{\mathcal{X}},$





$\rightsquigarrow$



$\forall E$ not toroidal

$$a(E, Y_i, B_i) \geq a(E, Y_i, \Delta_i) = a(E, X, \Sigma)$$

$$\geq 1 \geq a(E, X_i, B_i).$$

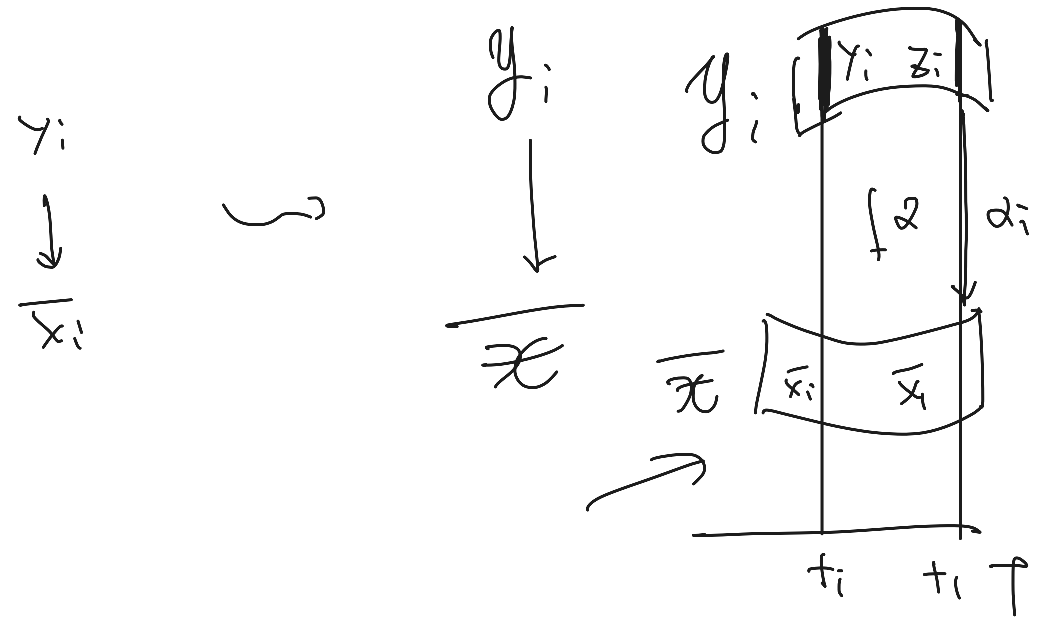
$\Rightarrow \quad p_*q^*(K_{Y_i}+B_i) \leq K_{X_i}+B_i$

$\Rightarrow \quad \mathrm{vol} = v.$

$X_i \dashrightarrow Y_i \rightsquigarrow (Y_i, B_{Y_i} + M_{Y_i})$ w/ $\mathrm{vol} = v_i := \mathrm{vol}(K_{X_i} + B_i + M_i)$

$\downarrow$

$\bar{X}_i$ $\swarrow$ seq of sm



$\exists\ \mathfrak{B}_i$

$\mathfrak{B}_i|_{Y_i} = B_{Y_i}$

take

$B_{Z_i} = \mathfrak{B}_i|_{Z_i}$

$M_{Z_i} = (\alpha^* \Sigma \mu_{ij} M_j)|_{Z_i}$

$= \alpha_i^* \Sigma \mu_{ij} \bar{M}_{ij}$

Claim. $\mathrm{vol}(K_{Z_i} + B_{Z_i} + M_{Z_i}) = \mathrm{vol}(K_{Y_i} + B_{Y_i} + M_{Z_i})$

Fix A ample/T on Y_i, $\forall l \in \mathbb{Z}_{>0}$

$$\left\{\begin{array}{l} M_{Z_i} + \frac{1}{l}A|_{Z_i} = (\alpha^* \sum \mu_j M_j + \frac{1}{l}A)|_{Z_i} \text{ ample} \\ M_{Y_i} + \frac{1}{l}A|_{Y_i} = (\alpha^* \sum \mu_j M_j + \frac{1}{l}A)|_{Y_i} \text{ ample} \end{array}\right.$$

$\Rightarrow \exists\, \Theta_l \ni t_1, t_i$ s.t.

$$\Theta_l \sim_{\mathbb{R}} \alpha^* \sum \mu_j M_j + \frac{1}{l}A \quad \text{ample} / U_l$$

(Y_i, Θ_l) lcg sm/ Θ_l.

HMX $\Rightarrow$ $\mathrm{vol}(K_{Y_i} + \Theta_l|_{Z_i}) = \mathrm{vol}(K_{Y_i} + \Theta_l|_{Y_i})$

Rem. for $\mathbb{R}$-div

$$\mathrm{vol}(K_{Z_i} + B_{Z_i} + M_{Z_i} + \tfrac{1}{l}A|_{Z_i}) = \mathrm{vol}(K_{Y_i} + B_{Y_i} + M_{Y_i} + \tfrac{1}{l}A|_{Y_i})$$

$l \nearrow +\infty \rightsquigarrow \mathrm{vol}(K_{Z_i} + B_{Z_i} + M_{Z_i}) = \mathrm{vol}(K_{Y_i} + B_{Y_i} + M_{Y_i})$

$(X_i, B_i + M_i) \rightsquigarrow (Z_i, B_{Z_i} + M_{Z_i})$

$$\downarrow f_i$$

$$(\overline{X}_i, \overline{\Sigma}_i) = (X, \Sigma)$$

Step 2 (= Proof of $Y_i \geq$)

- $B_{Z_i} \in \Phi$
- $M_{Z_i} = f_i^*(\Sigma \mu_{ij} M_j)$, M_j given on X
- $v_i = \mathrm{vol}(K_{Z_i} + B_{Z_i} + M_{Z_i}) \downarrow$
- $(f_i)_* B_{Z_i} \leq \Sigma$

$Z_i \xrightarrow{f_i} X$ on X

$C_i = f_{i*} B_{Z_i}$ $\quad C = \lim C_i$

$C \geq C_i$ $\quad$ $\boxed{f_i^*(K_X + C) \geq K_{Z_i} + B_{Z_i}}$ ($\geq \epsilon_i(i)$), $Z_i \to X$

Idea. $\forall t < 1 \nearrow 1$ s.t.

$$f_i^*(K_X + tC) \leq K_{Z_i} + B_{Z_i} \quad \text{for } i \gg 0.$$

if true,

$$\mathrm{vol}(K_X + C) \geq v_i = \mathrm{vol}(K_{Z_i} + B_{Z_i}) \geq \mathrm{vol}(K_X + tC)$$

$i \to \infty \downarrow$

$$v_0 \geq \mathrm{vol}(K_X + tC) \quad t \nearrow 1$$

$\rightsquigarrow \mathrm{vol}(K_X + C) = v_i \quad X.$ $\qquad \to \mathrm{vol}(K_X + C)$

Key. $g_i^*(K_X + tC) \le K_{Z_i} + B_{Z_i}$ for $i \gg 0$

Diff. ① D w/ $1 - a(D, X, tC) > \mu_D B_{Z_i}$ $\forall i$

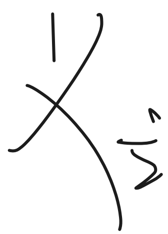
Def. b-div $\mathbf{B}_i = \begin{cases} B_{Z_i} & \text{on } Z_i \\ f_*^{-1} B_{Z_i} + E_X(f) & \forall f,\ Z \dashrightarrow Z_i \end{cases}$

b-div $\mathbf{C} = \lim \mathbf{B}_i$

$1 - a(D, X, C) > (\ge)\ \mu_D \mathbf{C}$

② $1 - a(D, X, C) = \mu_D \mathbf{C}$ but $\mathrm{Cent}_X D \not\subseteq \mathrm{Supp}\, C$.

③ $\{D \mid 1 - a(D, X, C) > \mu_D \mathbf{C}\}$ infinitely many



(Z_i, B_{Z_i})

$g_i \downarrow$

(X, Σ)

$$\underline{K_{Z_i} + B_{Z_i}} + (\overset{>0}{\ }) = g_i^*(K_X + C_i) \geq K_{Z_i} + B_{Z_i}$$

<u>def b-div</u>

$$\mathbb{B}_i = \begin{cases} B_{Z_i} & \text{on } Z_i \\ \phi_*^{-1} B_{Z_i} + \operatorname{Ex}(\phi), & \phi : Z \dashrightarrow Z_i \end{cases}$$

$\{0 < \mu_D \mathbb{B}_i < 1\}$ countable

---

def $\mathbb{C} = \lim \mathbb{B}_i$, well-defn.

$1 \in \underline{I} = \bar{I}$

$\forall D, \quad D \subseteq X, \quad \mu_D \mathbb{C} = \underline{\lim \mu_D \mathbb{B}_i}$

D exp'l/X if $\underline{D \subseteq X_i}^{\exists i}$ $\mu_D \mathbb{C} = \lim \mu_D \mathbb{B}_i$

D exp'l/X_i $^{\exists i}$ $\mu_D \mathbb{C} = \underline{1}$.

$(Z_i, B_{Z_i} + M_{Z_i})$

$g_i \downarrow$

fix $\to (X, \Sigma)$ (log sm)

- (X, Σ) str. tor. pair (2.12)
- $B_{Z_i} \in \mathcal{J}$, $g_{i*} B_{Z_i} \leq \Sigma$
- $M_{Z_i} = g_i^* \sum \mu_{ij} M_j$, $\mu_{ij} \in \mathcal{J}$,
- $v_i = \mathrm{vol}(K_{Z_i} + B_{Z_i} + M_{Z_i}) \downarrow$
- $g_i^*(K_X + C_i) \underset{(\geq)}{=} K_{Z_i} + B_{Z_i} + G_i$

Want. $t \in (0,1)$ $g_i^*(K_X + tC) \leq K_{Z_i} + B_{Z_i}$ for $i \gg 0$.

Defn. b-div $\mathbb{B}_i$ & $C'' = \lim \mathbb{B}_i$, $C = \lim C_i = C''_X$

Set.

- $\mathcal{D}_{\leq}(X, C) = \{ D$ exp'l toroidal (X,Σ) s.t. $\mu_D C'' \leq$ (w.r.t) $ta(D, X, C) \}$
- $\mathcal{D}_{<}(X, C)$ ($<$)
- $\mathcal{V}(X, C) = \{$ lc center V of (X, C) $\mid$ $V \cap \mathrm{Cent}_X D \neq \phi$, $\exists D \in \mathcal{D}_<(X, C) \}$

Def. $w = (\phi, r, l, d)$ on (X, C)

$\forall V$ lc center of (X, C), let $K_V + C_V = (K_X + C)|_V$

- $r = \mathrm{codim}_X V$, $l = \#\{ S$ exp'l$/V$ & $\mathrm{Cent}_V S \leq$ lc$(V, C_V) \}$

$d = \sum$ coeff of C_V let $w_V = (r, l, d)$

- $\underline{\phi = \# \mathcal{V}(X, C)} < +\infty$

if $\underline{\phi = 0}$, set $\underline{r = 0}$, $\underline{l = \# D_<(X, C)} < +\infty$, $\underline{d = \text{I coeff of } C}$

if $\underline{\phi > 0}$ choose $\underline{v \in \mathcal{V}(X, C)}$ w/

$$w_v = \max_{u \in \mathcal{V}(X,C)} \{ w_u \} = (r, l, d)$$

let $\underline{w := (\phi, r, l, d)}$ on (X, C).

Construction. $\underline{\forall D \in D_{\leq}(X, C)}$ $\quad ((h_i)^* B_{Z_i} + \Sigma_X(h_i))$

$W \xrightarrow{h_i} Z_i$, $\quad W \to Z_i'$, $\quad W \to X'$

$Z_i \to X$, $\quad Z_i' \to X'$, $\quad X \xleftarrow{\pi} X'$

$\underline{F_W + B_{i,W}'' \text{-MMP}/X'}$ w/ $\underline{\text{vol } v_i}$

$\underline{\pi^*(F_X + C) \geq F_{X'} + C'}$ $\quad C' = C_{X'}$ $\quad$ "=" $\Leftrightarrow D \notin D_<(X, C)$

why not log sm

extr D $\rightsquigarrow$ $\underline{D_{\leq}(X', C') \subsetneq D_{\leq}(X, C)}$

$\Rightarrow$ s.t. $\underline{(X', \Sigma')}$ str. toroidal $\quad \underline{D_<(X', C') \subseteq D_<(X, C)}$. $\quad$ "$\subsetneq$" $\Leftrightarrow \underline{D \in D_<(X, C)}$.

Fact. (Z_i, B_{Z_i}) $\quad$ data = original

If $(p, r, l) = 0$. $D_<(X, c) = \phi$

$\exists D \in D_\leq(X, c)$ $\quad Cut_X D \not\subseteq Supp\, C$ & $a(D, X, c) < 1$.

$\mu_P C' = 1 - a(D, X, c)$.

$p = 0$ $\quad a(D, X, tc)$

By Const. Extract all D. if $a(D, X, c) < 1$ $\in D_\leq(Y, c)$ $\Rightarrow Cut_X D \subseteq Supp\, C$.

Check. $g_i^*(K_X + tc) \leq K_{Z_i} + B_{Z_i}$ $(i \gg 0.)$

Want. $\exists X' \to X$ s.t. $w' = (0, 0, 0, d')$

$w \neq 0$. $(p, r, l) \neq 0$.

if $p = 0$, $r = 0$. $l \neq 0$. $l = \# D_<(X, C)$ $\exists D \in D_<(X, c)$

if $p > 0$, $r > 0$ $\Rightarrow \exists D \in D_<(X, C)$ s.t. $Cut_X D \cap V(Y, c) \neq \phi$

$$\begin{array}{ccc} Z_i & & Z_i \\ \downarrow & & \downarrow \\ X & \xleftarrow{f} & \underline{(X', C')} \end{array} \qquad D \in \mathcal{D}_{<}(X, c).$$

<u>Claim</u>. $w' <_D w$

$$\underline{f^*(K_X + c) \geq K_{X'} + C'}.$$

$$\mathcal{V}(X', C') \overset{\text{birtl}}{\dashrightarrow} \mathcal{V}(X, c)$$

$\Rightarrow \quad w' \leq w.$ $\qquad \nexists\, \mathcal{D}_{<}(X, c)$

if $p = 0 \Rightarrow p' = r' = 0 \Rightarrow l' < l \Rightarrow w' < w.$

if $p' < p \Rightarrow w' < w.$

<u>Assume</u>. $p' = p$, $\quad \underline{V} \in \mathcal{V}(X, c)$ s.t. min'l $\{ U \in \mathcal{V}(X, c) \mid C_X + D \cap U \neq \emptyset \}$

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \circlearrowleft & & \circlearrowleft \\ V' & \longrightarrow & V \end{array} \qquad K_X + c|_V = K_V + C_V$$

$$? \quad K_{X'} + c'|_{V'} = K_{V'} + C_{V'}.$$

Show. $w_{v'} < w_v$

$\quad\; \| \qquad\quad \|$

$(s', m', l') \quad (s, m, l)$

$s' = s$. $\quad m' \leq m$.

$f^{*}(Fx + c) \geq Fx' + c'$

$f_v^{*}(F_U + C_V) \supseteq F_{U'} + c'$.

"=" be basis of (V, Γ_V)

- if f_v contract div $\Rightarrow m' < m$. $\Rightarrow w_{v'} < w_v$.
- if f_v not contr, $m' = m \Rightarrow l' < l \Rightarrow w_{v'} < w_v$

Now show that $w' < w$

$\mathcal{V}(X', c') \qquad \mathcal{V}(X, c)$

$\quad \Psi \qquad\qquad\qquad \Psi$

$U' \longrightarrow U \cap \mathrm{Cent}_X D \neq \emptyset$.

if U min'l, $w_{U'} < w_U \leq (n, l, d)$.

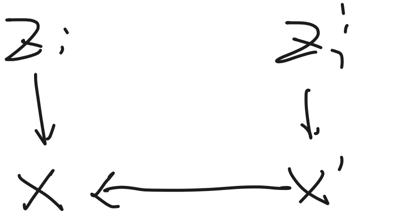
if U not min'l $\Rightarrow \exists T \in \mathcal{V}(X, c) \quad T \cap \mathrm{Cent}_X D = \phi$

& $U \subseteq T$.

$w_{U'} \leq w_U < w_T$

$\Rightarrow \quad \forall v' \in V(X', c') \qquad w_{v'} < (r, l, d)$

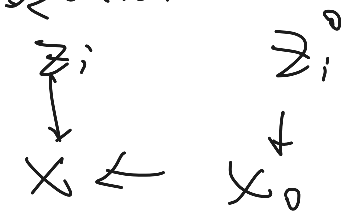
$\Rightarrow \quad w' < w.$



$w \downarrow \qquad \underline{(p, r, l, d)}$ J

$D \in D_<(X, c)$

⇝



$(p_0, r_0, l_0) = (0, 0, 0).$

$\square$