Contemporary Mathematics

Chapter 4. Effective birationality

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1. Introduction

In this chapter, we consider the effective birationality of anti-pluri-canonical systems of ϵ-lc Fano varieties, i.e. [**1**, Theorem 1.2]. Instead of proving [**1**, Theorem 1.2], we will prove two special cases of [**1**, Theorem 1.2] under additional assumptions, which are crucial for the proof of other main theorems. The main results are Propositions 4.2 and 5.1. All contents in this chapter are based on [**1**, Section 4].

We briefly explain the strategy of showing the effective birationality. Given an ϵ-lc Fano variety X of dimension d, consider $m \in \mathbb{N}$ to be the smallest number such that $|-mK_X|$ defines a birational map. The goal is to show that m is bounded from above. One important idea is that, in order to show that m is bounded from above, we first show that $\frac{m}{n}$ is bounded from above, where $n \in \mathbb{N}$ is the smallest number such that $\mathrm{vol}(-nK_X) > (2d)^d$.

Once $\frac{m}{n}$ is bounded from above, then $\mathrm{vol}(-mK_X)$ is bounded from above. Since $|-mK_X|$ defines a birational map, this implies that X is birationally bounded, and we can then construct a nice log bounded family $\overline{\mathcal{P}}$, see Lemma 3.3. Now we can work on the log bounded family $\overline{\mathcal{P}}$. From the assumption that X is ϵ-lc, we can construct a sub-ϵ-lc pair $(\overline{W}, \Lambda_{\overline{W}})$ whose support is in $\overline{\mathcal{P}}$. On the other hand, under some additional assumptions, we may construct an effective $\mathbb{Q}$-divisor L such that $(\overline{W}, \Lambda_{\overline{W}} + L)$ is not sub-klt. By comparing the singularities of these two pairs, we know that numerically L cannot be too small, see Proposition 2.1. On the other hand, by the construction of L, we know that L can be arbitrarily small as long as m is unbounded, which shows the boundedness of m.

Then it remains to show that $\frac{m}{n}$ is bounded from above. The idea is to construct isolated non-klt centers by $-nK_X$, that is, for a general point $x \in X$, we need to construct an effective $\mathbb{Q}$-divisor $\Delta \sim_{\mathbb{Q}} -knK_X$ where k is independent of X, so that (X, Δ) has an isolated non-klt center at x. Once this is done, then by using vanishing theorem, it is easy to see that $|-2knK_X|$ defines a birational map and we can then conclude that $\frac{m}{n} \leq 2k$. From the definition of n, it is easy to construct an effective $\mathbb{Q}$-divisor $\Delta \sim_{\mathbb{Q}} -nK_X$, so that (X, Δ) has a non-klt center G containing x. The problem here is that the dimension of G could be positive. If $\mathrm{vol}(-mK_X|_G)$ is bounded from below, then it is easy to construct a new non-klt center with

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dimension strictly smaller than G by standard methods, and after finitely many steps, we get isolated non-klt centers.[1.1] So finally we need to consider the case that $\mathrm{vol}(-mK_X|_G)$ is not bounded from below, which in fact implies that $\mathrm{vol}(-mK_X|_G)$ is bounded from above. Along with the fact that $|-mK_X|_G|$ defines a birational map on G as x is general, such G is log birationally bounded, and we can then construct a nice log bounded family $\overline{\mathcal{P}}'$ by applying Lemma 3.1. Similarly to the proof of boundedness of m, we can work on the log bounded family $\overline{\mathcal{P}}'$. From the assumption that X is ϵ-lc, we can construct a sub-ϵ-lc pair $(\overline{F}, \Lambda_{\overline{F}})$ whose support is in $\overline{\mathcal{P}}'$. On the other hand, under some additional assumptions, we may construct an effective $\mathbb{Q}$-divisor L' such that $(\overline{F}, \Lambda_{\overline{F}} + L')$ is not sub-klt. Here note that we work on $\overline{F}$, which is a birational model of G. In order to consider the singularities of $\overline{F}$, we need to consider the singularities of G which is induced from that of X. This requires a nice adjunction theory, which is highly non-trivial, see [**1**, Section 3] or Chapter 9. By comparing the singularities of these two pairs, we know that numerically L' cannot be too small, see Proposition 2.1. On the other hand, by the construction of L', we know that L' can be arbitrarily small as long as $\frac{m}{n}$ is unbounded, which shows the boundedness of $\frac{m}{n}$.

2. Singularities in bounded families

Proposition 2.1. Let $\epsilon \in \mathbb{R}_{>0}$ and let $\mathcal{P}$ be a bounded set of couples. Then there is $\lambda \in \mathbb{R}_{>0}$ depending only on $\epsilon, \mathcal{P}$ satisfying the following. Let (X, B) be a projective sub-pair and let T be a reduced divisor on X. Assume

- (X, B) is sub-ϵ-lc and $(X, \mathrm{Supp}(B^{>0} + T)) \in \mathcal{P}$;
- L is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X;
- $L \sim_{\mathbb{R}} \tilde{L}$ for some $\mathbb{R}$-divisor $\tilde{L}$ on X;
- $\mathrm{Supp}(\tilde{L}^{>0}) \subset T$, and the coefficients of $\tilde{L}$ are at most λ.

Then $(X, B + L)$ is sub-klt.

Remark 2.2. *Proposition 2.1 here is slightly more generalized than* [**1**, Proposition 4.2]. *The difference is that we do not assume that B is effective. This turns out to be very useful in applications.*

Proof of Proposition 2.1. Since $\mathcal{P}$ is a bounded set of couples, we can find a log resolution $\phi : W \to X$ of $(X, \mathrm{Supp}(B^{>0} + T))$ and write

$$K_W + B_W = \phi^*(K_X + B) + E,$$

such that $(W, \mathrm{Supp}(B_W + T_W))$ belongs to a bounded set of couples depending only on $\mathcal{P}$, where $B_W \geq 0$ and $E \geq 0$ have no common components, and T_W is the sum of the birational transform of T and all ϕ-exceptional divisors. Now (W, B_W) is ϵ-lc. Let $L_W = \phi^* L$, and $\tilde{L}_W = \phi^* \tilde{L}$. Then there is an integer $m > 0$ depending only on $\mathcal{P}$ so that the coefficients of $\tilde{L}_W^{>0}$ is at most $m\lambda$. It suffices to prove that $(W, B_W + L_W)$ is klt. By the boundedness there exists a very ample divisor H_W

[1.1] This method is also used to show the effective birationality of pluricanonical systems on algebraic varieties of general type, see [**2**], [**5**], [**6**], [**4**]. The difference is that for varieties of general type, the center G is again of general type since x is general, and one may apply induction on G to conclude that $\mathrm{vol}(mK_X|_G)$ is indeed bounded from below. In our case, this induction step fails, so we need further discussions.

on W such that, $T_W \cdot H_W^{d-1} < M$ for some number $M > 0$ depending only on $\mathcal{P}$. Let $\lambda = \frac{\epsilon}{mM}$, then for any point $w \in W$, we have

$$\mathrm{mult}_w L_W \leq L_W \cdot H_W^{d-1} = \tilde{L}_W \cdot H_W^{d-1} \leq m\lambda T_W \cdot H_W^{d-1} < \epsilon.$$

By Exercise 6.2, $(W, B_W + L_W)$ is klt. □

3. Construction of bounded families

LEMMA 3.1. Let $d \in \mathbb{N}$ and $v_1, v_2 \in \mathbb{R}_{>0}$. Let $\mathcal{P}$ be a set of (Y, C, D) satisfying the following:

- Y is a normal projective variety of dimension d, C is a (possibly zero) reduced integral divisor on Y, and D is an effective nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y;
- there exists a log resolution $\phi : Z \to Y$ and a base point free divisor H_Z on Z such that $|H_Z|$ defines a birational map and $\phi^* D \geq H_Z$;
- write Σ_Z to be the support of $\phi^*(C + D)$ and all ϕ-exceptional divisors, then
$$\mathrm{vol}(K_Z + \Sigma_Z + 2(2d + 1)H_Z) \leq v_1;$$
- $\mathrm{vol}(D) \leq v_2$.

Then $\mathcal{P}$ is log birationally bounded. More precisely, there exists a log bounded family $\overline{\mathcal{P}}$ of couples such that for each $(Y, C, D) \in \mathcal{P}$, there exists a couple $(\overline{Z}, \Sigma_{\overline{Z}}) \in \overline{\mathcal{P}}$ satisfying the following:

- Y is birational to $\overline{Z}$, $(\overline{Z}, \Sigma_{\overline{Z}})$ is log smooth, where we may take a higher model of Z such that the induced map $\psi : Z \dashrightarrow \overline{Z}$ is a morphism;
- $\Sigma_{\overline{Z}}$ consists of the support of the strict transform of $C + D$ and divisors exceptional over Y;
- the coefficients of $\psi_* \phi^* D$ are bounded from above by a number depending only on d, v_1, and v_2.

PROOF. This is just [**3**, Lemmas 3.2 and 2.4.2(4)]. To be more precise, consider the birational morphism $Z \to \tilde{Z}$ defined by $|H_Z|$, then $(\tilde{Z}, \Sigma_{\tilde{Z}})$ is log bounded, where $\Sigma_{\tilde{Z}}$ is the pushforward of Σ_Z. Taking a log resolution of $(\tilde{Z}, \Sigma_{\tilde{Z}})$, we get a log smooth couple $(\overline{Z}, \Sigma_{\overline{Z}})$ which is still in a log bounded family, say $\overline{\mathcal{P}}$, where $\Sigma_{\overline{Z}}$ is the sum of the strict transform of $\Sigma_{\tilde{Z}}$ and all exceptional divisors over $\tilde{Z}$, in other words, $\Sigma_{\overline{Z}}$ consists of the support of strict transform of $C + D$ and divisors exceptional over Y. We may take a higher model of Z such that the induced map $\psi : Z \dashrightarrow \overline{Z}$ is a morphism. By the boundedness, there exists a very ample divisor $H_{\overline{Z}}$ on $\overline{Z}$ and a number b depending only on $\overline{\mathcal{P}}$ such that $b\psi_* H_Z - H_{\overline{Z}}$ is big. Recall that $\psi^* \psi_* H_Z = H_Z$ by the construction. Now the coefficients of $\psi_* \phi^* D$ are bounded by the following intersection number:

$$\begin{aligned} \psi_* \phi^* D \cdot H_{\overline{Z}}^{d-1} &= \phi^* D \cdot \psi^* H_{\overline{Z}}^{d-1} \\ &\leq \mathrm{vol}(\phi^* D + \psi^* H_{\overline{Z}}) \\ &\leq \mathrm{vol}(\phi^* D + bH_Z) \\ &\leq \mathrm{vol}((1 + b)\phi^* D) \\ &\leq (1 + b)^d v_2. \end{aligned}$$

Here the first inequality holds since both D and $H_{\overline{Z}}$ are nef. □

NOTATION 3.2. *Let X be a klt Fano variety and $m \in \mathbb{N}$ such that $|-mK_X|$ defines a birational map. Then we may take a resolution $\phi : W \to X$ such that $\phi^*(-mK_X) \sim A_W + R_W$ where A_W is base point free and R_W is the fixed part of $\phi^*(-mK_X)$. We may pick an A_W general in its linear system. Denote $\Delta_m := \phi_* A_W + \phi_* R_W \sim -mK_X$. Here we remark that R_W is in general a $\mathbb{Q}$-divisor, but $\phi_* R_W$ is an integral Weil divisor.*

LEMMA 3.3. Let $d \in \mathbb{N}$ and $\epsilon, v \in \mathbb{R}_{>0}$. Let $\mathcal{P}$ be a set of varieties such that for each $X \in \mathcal{P}$, the following hold:

- X is an ϵ-lc Fano variety of dimension d;
- $n > 1$, and $\frac{m}{n} < v$ where we denote by $m \in \mathbb{N}$ the smallest number such that $|-mK_X|$ defines a birational map, and $n \in \mathbb{N}$ the smallest number such that $\mathrm{vol}(-nK_X) > (2d)^d$.

Then there exists a log bounded family $\overline{\mathcal{P}}$ of couples such that for each $X \in \mathcal{P}$, there exists a couple $(\overline{W}, \Sigma_{\overline{W}}) \in \overline{\mathcal{P}}$ satisfying the following:

- X is birational to $\overline{W}$, $(\overline{W}, \Sigma_{\overline{W}})$ is log smooth, where we may take a higher model of W such that the induced map $\psi : W \dashrightarrow \overline{W}$ is a morphism, where $\phi : W \to X$ satisfies the assumptions in Notation 3.2;
- $\Sigma_{\overline{W}}$ consists of the support of the strict transform of Δ_m and divisors exceptional over X where Δ_m is defined in Notation 3.2;
- the coefficients of $\psi_*\phi^*\Delta_m$ are bounded from above by a number depending only on d, ϵ, and v.

PROOF. By the assumption, $|-mK_X|$ defines a birational map. Take $\phi : W \to X$, A_W, R_W, Δ_m as in Notation 3.2. By the minimality of n, $\mathrm{vol}(-(n-1)K_X) \le (2d)^d$. Hence

$$\begin{aligned}\mathrm{vol}(-mK_X) &= \left(\frac{m}{n-1}\right)^d \mathrm{vol}(-(n-1)K_X)\\ &\le \left(\frac{2m}{n}\right)^d \mathrm{vol}(-(n-1)K_X)\\ &\le (4vd)^d.\end{aligned}$$

Write Σ_W to be the support of $A_W + R_W$ and all ϕ-exceptional divisors, then

$$\begin{aligned}&\mathrm{vol}(K_W + \Sigma_W + 2(2d+1)A_W)\\ \le&\,\mathrm{vol}(K_X + \phi_*\Sigma_W + 2(2d+1)\phi_*A_W)\\ \le&\,\mathrm{vol}(K_X + \phi_*(A_W + R_W) + 2(2d+1)\phi_*A_W)\\ \le&(4d+3)^d\,\mathrm{vol}(\phi_*R_W + \phi_*A_W)\\ =&(4d+3)^d\,\mathrm{vol}(-mK_X)\\ \le&(4vd(4d+3))^d.\end{aligned}$$

Now we may apply Lemma 3.1 to $(Y, C, D, Z, H_Z) = (X, 0, \Delta_m, W, A_W)$ to finish the proof. $\square$

4. Effective birationality for Fano varieties with good $\mathbb{Q}$-complements

PROPOSITION 4.1. ([**1**, Proposition 4.8]) Let $d \in \mathbb{N}$ and $\epsilon, \delta \in \mathbb{R}_{>0}$. Then there exists a number v depending only on d, ϵ, and δ satisfying the following. Assume

- X is an ϵ-lc Fano variety of dimension d;

- $m \in \mathbb{N}$ is the smallest number such that $|-mK_X|$ defines a birational map;
- $n \in \mathbb{N}$ is a number such that $\mathrm{vol}(-nK_X) > (2d)^d$; and
- $nK_X + N \sim_{\mathbb{Q}} 0$ for some $\mathbb{Q}$-divisor N with coefficients $\geq \delta$.

Then $\frac{m}{n} < v$.

Proof. **Step 1.** Construct a family of non-klt centers with bounded volumes.

By [**1**, 2.31(2)] (see Section 1 of Chapter 9), since $\mathrm{vol}(-nK_X) > (2d)^d$, there is a bounded family of subvarieties of X such that for two general points x, y in X, there is a member G of the family and an effective $\mathbb{Q}$-divisor $\Delta \sim_{\mathbb{Q}} -(n+1)K_X$ such that (X, Δ) is lc near x with a unique non-klt place whose center contains x, that center is G, and (X, Δ) is not klt at y. Recall that this family is given by finitely many morphisms $V^j \to T^j$ of projective varieties with surjective morphisms $V^j \to X$ and G is a general fiber of one of $V^j \to T^j$. Denote $k := \max\{\dim V^j - \dim T^j\}$. We do induction on k.

If $k = 0$, that is, $\dim G = 0$ for all general G, then $-(n+2)K_X$ is potentially birational, hence $|K_X - (n+2)K_X|$ defines a birational map by [**3**, Lemma 2.3.4]. By the minimality of m, $m \leq n+1 \leq 2n$, which implies that $\frac{m}{n} \leq 2$.

Now assume that $k > 0$. Define $l \in \mathbb{N}$ to be the smallest number such that $\mathrm{vol}(-lK_X|_G) > d^d$ for all general G with $\dim G > 0$. Then there exists j such that if G is a general fiber of $V^j \to T^j$, then $\dim G > 0$ and $\mathrm{vol}(-(l-1)K_X|_G) \leq d^d$. Now since $\mathrm{vol}(-lK_X|_G) > d^d$, by [**1**, 2.31(2)], after replacing n by $n + (d-1)l$, we may construct a new bounded family of subvarieties of X corresponding to non-klt centers of $(X, \Delta' \sim_{\mathbb{Q}} -(n + (d-1)l + 1)K_X)$ while k is strictly decreased. Hence by the induction, there exists a number v' depending only on d, ϵ, and δ such that $\frac{m}{n+(d-1)l+1} < v'$. If $m > 2v'(d-1)l$, then

$$2m < 2v'(n + (d-1)l + 1) < 2v'n + m + 2v'$$

which implies that $m < 4v'n$ and we are done.

Hence from now on, we may assume that $m \leq 2v'(d-1)l$. If $l = 1$, then $m \leq 2v'(d-1) \leq 2v'(d-1)n$ and again we are done. Hence we may assume that $l > 1$ and therefore $m \leq 4v'd(l-1)$. In this case, we have

$$\mathrm{vol}(-mK_X|_G) \leq (4v'd)^k \,\mathrm{vol}(-(l-1)K_X|_G) \leq (4v'd)^k d^d$$

if G is a general fiber of $V^j \to T^j$.

In summary, we constructed a family $V^j \to T^j$ of projective varieties with a surjective morphism $V^j \to X$ such that if G is a general fiber of $V^j \to T^j$, then $\dim G = k > 0$, there exists an effective $\mathbb{Q}$-divisor $\Delta \sim_{\mathbb{Q}} -(n+1)K_X$ and there is a unique non-klt place of (X, Δ) whose center is G, and $\mathrm{vol}(-mK_X|_G) \leq v_2$ for a number v_2 depending only on d, ϵ, and δ.

Step 2. Construct a bounded family.

Take F to be the normalization of G. By [**1**, Theorem 3.10] (see Theorem 2.4 of Chapter 9) and ACC for LCT [**4**, Theorem 1.1], there is an effective $\mathbb{Q}$-divisor Θ_F with coefficients in a DCC set Φ depending only on d such that we may write

$$(K_X + \Delta)|_F = K_F + \Delta_F = K_F + \Theta_F + P_F$$

where P_F is pseudo-effective. Pick a general ample divisor $H' \sim_{\mathbb{Q}} -nK_X$ with sufficiently small coefficients, after replacing n by $2n$, Δ by $\Delta + H'$, and P_F by

$P_F + H'|_F$, we may assume that P_F is effective and big. Since G is general, by [**1**, Lemma 3.12] (see Theorem 2.5 of Chapter 9), we may write $K_X|_F = K_F + \Lambda_F$ for some sub-boundary Λ_F such that (F, Λ_F) is sub-ϵ-lc and $\Lambda_F \leq \Theta_F \leq \Delta_F$.

By the assumption, $|-mK_X|$ defines a birational map. Take $\phi : W \to X$, A_W, R_W, Δ_m as in Notation 3.2. Take a log resolution $f : F' \to F$ of (F, Δ_F) such that the induced map $F' \dashrightarrow W$ (which is well-defined since G is general) is a morphism. Denote $A_{F'} := A_W|_{F'}$ which is base point free and defines a birational map on F'. Denote $M_F := \Delta_m|_F$. Note that $f^*M_F = (A_W + R_W)|_{F'} \geq A_{F'}$.

Take $\Sigma_{F'}$ to be sum of the strict transforms of M_F, $\operatorname{Supp}\Theta_F$, and f-exceptional divisors. Fix a rational number $\epsilon' \in (0, \epsilon)$ such that $\epsilon' < \min \Phi^{>0}$. By the definition of Φ, $\operatorname{Supp}(\Theta_F) \leq \frac{\Theta_F}{\epsilon'}$. Note that by [**1**, Lemma 3.11], $\operatorname{Supp}(M_F) \leq \Theta_F + M_F$ since Δ_m is an integral Weil divisor. Recall that by [**1**, Lemma 2.46] (see Lemma 3.10 of Chapter 2), $K_{F'} + (2k+1)A_{F'}$ is big. Hence

$$
\begin{aligned}
&\operatorname{vol}(K_{F'} + \Sigma_{F'} + 2(2k+1)A_{F'})\\
\leq{}&\operatorname{vol}(K_{F'} + \Sigma_{F'} + 2(2k+1)A_{F'} + {\epsilon'}^{-1}(K_{F'} + (2k+1)A_{F'}))\\
\leq{}&\operatorname{vol}(K_F + \Sigma_F + 2(2k+1)A_F + {\epsilon'}^{-1}(K_F + (2k+1)A_F))\\
\leq{}&\operatorname{vol}((1+{\epsilon'}^{-1})K_F + \operatorname{Supp}(M_F) + \operatorname{Supp}(\Theta_F) + (2+{\epsilon'}^{-1})(2k+1)A_F)\\
\leq{}&\operatorname{vol}((1+{\epsilon'}^{-1})K_F + \Theta_F + M_F + {\epsilon'}^{-1}\Theta_F + (2+{\epsilon'}^{-1})(2k+1)A_F)\\
\leq{}&\operatorname{vol}((1+{\epsilon'}^{-1})(K_F + \Theta_F + P_F) + m(2+{\epsilon'}^{-1})(2k+2)(-K_X|_F))\\
\leq{}&\operatorname{vol}(((1+{\epsilon'}^{-1})n + (2+{\epsilon'}^{-1})(2k+2)m)(-K_X|_F)).
\end{aligned}
$$

We may always assume that $n \leq m$ otherwise there is nothing to prove. Hence, by Step 1, $\operatorname{vol}(K_{F'} + \Sigma_{F'} + 2(2d+1)A_{F'}) \leq v_1$ for a number v_1 depending only on d, ϵ, and δ.

Now we may apply Lemma 3.1 to

$$(Y, C, D, Z, H_Z) = (F, \operatorname{Supp}\Theta_F, M_F, F', A_{F'})$$

to construct a log bounded family $\overline{\mathcal{P}}$ of couples such that there exists a couple $(\overline{F}, \Sigma_{\overline{F}}) \in \overline{\mathcal{P}}$ satisfying the following:

- F is birational to $\overline{F}$, $(\overline{F}, \Sigma_{\overline{F}})$ is log smooth;
- $\Sigma_{\overline{F}}$ consists of the support of $M_{\overline{F}} + \Theta_{\overline{F}}$ and divisors exceptional over F;
- the coefficients of $M_{\overline{F}}$ are bounded from above by a number, say u, depending only on d, ϵ, and δ.

Here we may take a higher model of F' such that the induced map $g : F' \dashrightarrow \overline{F}$ is a morphism, $M_{\overline{F}} := g_*f^*M_F$, and $\Theta_{\overline{F}} := g_*f^*\Theta_F$.

Step 3. Apply Proposition 2.1.

Write

$$
\begin{aligned}
K_{\overline{F}} + \Lambda_{\overline{F}} &:= g_*f^*(K_F + \Lambda_F),\\
K_{\overline{F}} + \Delta_{\overline{F}} &:= g_*f^*(K_F + \Delta_F).
\end{aligned}
$$

Since $\Lambda_F \leq \Theta_F$, $\operatorname{Supp}(\Lambda_{\overline{F}}^{\geq 0}) \subset \Sigma_{\overline{F}}$ by the construction. Moreover, the coefficients of $\Lambda_{\overline{F}}$ is at most $1 - \epsilon$ since (F, Λ_F) is sub-ϵ-lc. Hence, $(\overline{F}, \Lambda_{\overline{F}})$ is again sub-ϵ-lc.

Consider $J_F = \frac{1}{\delta}N|_F$ and let D be a component of J_F. Then by [**1**, Lemma 3.11],

$$\mu_D(\Delta_F + J_F) \geq \mu_D(\Theta_F + J_F) \geq 1.$$

In particular, $(F, \Delta_F + J_F)$ is not sub-klt. Note that $K_F + \Delta_F + J_F \sim_{\mathbb{Q}} -(1 + \frac{1}{\delta})nK_X|_F$ is nef. Hence $(\overline{F}, \Delta_{\overline{F}} + g_*f^*J_F)$ is not sub-klt by Exercise 6.4. Rewrite this sub-pair as $(\overline{F}, \Lambda_{\overline{F}} + \Delta_{\overline{F}} - \Lambda_{\overline{F}} + g_*f^*J_F)$. Applying Proposition 2.1 to the sub-pair $(\overline{F}, \Lambda_{\overline{F}})$, $L = \Delta_{\overline{F}} - \Lambda_{\overline{F}} + g_*f^*J_F$ and $\tilde{L} = (\frac{n+1}{m} + \frac{n}{m\delta})M_{\overline{F}} \sim_{\mathbb{Q}} L$, there is $\lambda \in \mathbb{R}_{>0}$ depending only on $\epsilon, \overline{\mathcal{P}}$ such that $u(\frac{n+1}{m} + \frac{n}{m\delta}) > \lambda$, which implies that $\frac{m}{n} < \frac{u(2+\delta)}{\lambda\delta}$. □

Proposition 4.2. ([**1**, Proposition 4.9]) *Let $d \in \mathbb{N}$ and $\epsilon, \delta \in \mathbb{R}_{>0}$. Then there exists a number $m \in \mathbb{N}$ depending only on d, ϵ, and δ satisfying the following. Assume that X is an ϵ-lc Fano variety of dimension d such that $K_X + B \sim_{\mathbb{Q}} 0$ for some $\mathbb{Q}$-divisor B with coefficients $\geq \delta$. Then $|-mK_X|$ defines a birational map.*

Proof. Take $m \in \mathbb{N}$ the smallest number such that $|-mK_X|$ defines a birational map, and $n \in \mathbb{N}$ the smallest number such that $\mathrm{vol}(-nK_X) > (2d)^d$. By the assumption, the assumptions of Proposition 4.1 are satisfied, hence there exists a number v depending only on d, ϵ, and δ such that $m/n < v$. We may assume that $n > 1$, otherwise $m < v$ and there is nothing to prove. Now the assumptions of Lemma 3.3 are satisfied. Hence there exists a log bounded family $\overline{\mathcal{P}}$ of couples such that there exists a couple $(\overline{W}, \Sigma_{\overline{W}}) \in \overline{\mathcal{P}}$ satisfying the following:

- X is birational to $\overline{W}$, $(\overline{W}, \Sigma_{\overline{W}})$ is log smooth, where we may take a higher model of W such that the induced map $\psi : W \dashrightarrow \overline{W}$ is a morphism, where $\phi : W \to X$ satisfies the assumptions in Notation 3.2;
- $\Sigma_{\overline{W}}$ consists of the support of the strict transform of Δ_m and divisors exceptional over X where Δ_m is defined in Notation 3.2;
- the coefficients of $\psi_*\phi^*\Delta_m$ are bounded from above by a number, say u, depending only on d, ϵ, and v.

Write $K_{\overline{W}} + \Lambda_{\overline{W}} := \psi_*\phi^*K_X$. Note that $\mathrm{Supp}(\Lambda_{\overline{W}}) \subset \Sigma_{\overline{W}}$ by the construction since $\Lambda_{\overline{W}}$ is exceptional over X. Since X is ϵ-lc, the coefficients of $\Lambda_{\overline{W}}$ are at most $1-\epsilon$. Hence $(\overline{W}, \Lambda_{\overline{W}})$ is sub-ϵ-lc.

By the assumption, the coefficients of $\frac{1}{\delta}B$ is at least 1, hence $(X, \frac{1}{\delta}B)$ is not klt. Also note that $K_X + \frac{1}{\delta}B \sim_{\mathbb{Q}} -\frac{1-\delta}{\delta}K_X$ is ample where without loss of generality we may assume that $\delta < 1$. Hence $(\overline{W}, \Lambda_{\overline{W}} + \frac{1}{\delta}\psi_*\phi^*B)$ is not sub-klt by Exercise 6.4. Note that $\frac{1}{\delta}\psi_*\phi^*B \sim_{\mathbb{Q}} \frac{1}{m\delta}\psi_*\phi^*(\Delta_m)$. Applying Proposition 2.1 to the sub-pair $(\overline{W}, \Lambda_{\overline{W}})$, $L = \frac{1}{\delta}\psi_*\phi^*B$ and $\tilde{L} = \frac{1}{m\delta}\psi_*\phi^*(\Delta_m)$, there is $\lambda \in \mathbb{R}_{>0}$ depending only on $\epsilon, \overline{\mathcal{P}}$ such that $\frac{u}{m\delta} > \lambda$, which means that $m < \frac{u}{\lambda\delta}$. □

5. Effective birationality for nearly canonical Fano varieties

Proposition 5.1. ([**1**, Proposition 4.11]) *Let $d \in \mathbb{N}$. Then there exist numbers $\tau \in (0, 1)$ and $m \in \mathbb{N}$ depending only on d satisfying the following. If X is a τ-lc Fano variety of dimension d, then $|-mK_X|$ defines a birational map.*

Proof. Fix $\tau \in (\frac{1}{2}, 1)$ which is sufficiently close to 1. (We may get the restriction on τ which depending only on d during the proof and state them in the end). Take $m \in \mathbb{N}$ the smallest number such that $|-mK_X|$ defines a birational map, and $n \in \mathbb{N}$ the smallest number such that $\mathrm{vol}(-nK_X) > (2d)^d$.

Step 1. Similar to Proposition 4.1, we show that there exists a number v depending only on d such that $m/n < v$.

Step 1.1. Construct a bounded family.

Note that we may apply Steps 1 and 2 of proof of Proposition 4.1 here (take $\epsilon = \frac{1}{2}$ and ignore δ). We will keep the notation and constructions from there. Recall that we can construct a family $V^j \to T^j$ of projective varieties with a surjective morphism $V^j \to X$ such that if G is a general fiber of $V^j \to T^j$, then $\dim G = k > 0$, there exists an effective $\mathbb{Q}$-divisor $\Delta \sim_{\mathbb{Q}} -(n+1)K_X$ and there is a unique non-klt place of (X, Δ) whose center is G, and $\mathrm{vol}(-mK_X|_G) \leq v_2$ for a number v_2 depending only on d. Take F to be the normalization of G. Then we can construct a log bounded family $\overline{\mathcal{P}}$ of couples such that there exists a couple $(\overline{F}, \Sigma_{\overline{F}}) \in \overline{\mathcal{P}}$ satisfying the following:

- F is birational to $\overline{F}$, $(\overline{F}, \Sigma_{\overline{F}})$ is log smooth;
- $\Sigma_{\overline{F}}$ consists of the support of $M_{\overline{F}} + \Theta_{\overline{F}}$ and divisors exceptional over F;
- the coefficients of $M_{\overline{F}}$ are bounded from above by a number, say u, depending only on d.

Here we may take a higher model of F' such that the induced map $g : F' \dashrightarrow \overline{F}$ is a morphism, $M_{\overline{F}} := g_* f^* M_F$, and $\Theta_{\overline{F}} := g_* f^* \Theta_F$.

Recall that by the construction, (F, Λ_F) is sub-τ-lc. Recall that by the construction, $\Lambda_F \leq \Theta_F \leq \Delta_F$ and the coefficients of Θ_F belong to a DCC set Φ depending only on d. Denote $\min \Phi^{>0} = b$. We may assume that $\tau > 1 - \frac{b}{4}$ and fix $\tau' = 1 - \frac{b}{2}$.

Step 1.2. Reduce to the case that (F, Δ_F) is τ'-lc and $\Lambda_F \leq \Theta_F = 0$.

To the contrary, if (F, Δ_F) is not τ'-lc, then $(F, \Delta_F + c(\Delta_F - \Lambda_F))$ is not lc for $c = \frac{\tau'}{\tau - \tau'} \leq \frac{2\tau'}{b}$ by the discrepancy computation. Since $K_F + \Delta_F + c(\Delta_F - \Lambda_F) \sim_{\mathbb{Q}} -(n + c(n+1))K_X|_F$ is nef, $(\overline{F}, \Delta_{\overline{F}} + c(\Delta_{\overline{F}} - \Lambda_{\overline{F}}))$ is not klt by Exercise 6.4 where $\Lambda_{\overline{F}}$ is defined by $K_{\overline{F}} + \Lambda_{\overline{F}} = g_* f^*(K_F + \Lambda_F)$. Rewrite this sub-pair as $(\overline{F}, \Lambda_{\overline{F}} + (1+c)(\Delta_{\overline{F}} - \Lambda_{\overline{F}}))$. Recall that $(\overline{F}, \Lambda_{\overline{F}})$ is a sub-τ-lc (which is also sub-$\frac{1}{2}$-lc) pair with $\mathrm{Supp}(\Lambda_{\overline{F}}^{\geq 0}) \subset \Sigma_{\overline{F}}$. Hence we may apply Proposition 2.1 to $L = (1+c)(\Delta_{\overline{F}} - \Lambda_{\overline{F}})$ and $\tilde{L} = \frac{(c+1)(n+1)}{m} M_{\overline{F}} \sim_{\mathbb{Q}} L$, there is $\lambda \in \mathbb{R}_{>0}$ depending only on $\overline{\mathcal{P}}$ such that $u\frac{(c+1)(n+1)}{m} > \lambda$, which implies that $\frac{m}{n} < \frac{2u(1+c)}{\lambda}$ which is a number depending only on d. Hence in this case we can complete Step 1. So we may assume that $(F, \Delta_F = \Theta_F + P_F)$ is τ'-lc for any choice of $P_F \geq 0$ from now on.

Note that by the construction, $\Lambda_F \leq \Theta_F \leq \Delta_F$ and the coefficients of Θ_F belong to a DCC set Φ depending only on d. On the other hand, the coefficients of Δ_F are at most $1 - \tau'$. Then the coefficients of Θ_F are at most $1 - \tau' < b = \min \Phi^{>0}$, which implies that $\Lambda_F \leq \Theta_F = 0$.

Step 1.3. Reduce to the case that $K_{\overline{F}}$ is pseudo-effective with $\kappa_\sigma(K_{\overline{F}}) = 0$.

Assume that $K_{\overline{F}}$ is not pseudo-effective, by the boundedness of $\overline{\mathcal{P}}$ and [**1**, Lemma 2.35] (see Lemma 4.8 of Chapter 2), there exists a number λ' depending only on $\overline{\mathcal{P}}$ such that $K_{\overline{F}} + \lambda' \Sigma_{\overline{F}}$ is not pseudo-effective. By the construction,

$$K_{\overline{F}} + \Lambda_{\overline{F}} + \frac{1}{m} M_{\overline{F}} \sim_{\mathbb{Q}} 0.$$

In particular, $K_{\overline{F}} + \Lambda_{\overline{F}}^{\geq 0} + \frac{1}{m} M_{\overline{F}}$ is pseudo-effective. Note that the support of $\Lambda_{\overline{F}}^{\geq 0} + \frac{1}{m} M_{\overline{F}}$ is contained in $\Sigma_{\overline{F}}$ by the construction and its coefficients are at most $1 - \tau + \frac{u}{m}$. Hence $1 - \tau + \frac{u}{m} > \lambda'$. We may assume that $\tau > 1 - \frac{\lambda'}{2}$, which implies that $m \leq \frac{2u}{\lambda'}$ and we are done in this case.

Hence we may assume that $K_{\overline{F}}$ is pseudo-effective from now on.

Assume that $\kappa_\sigma(K_{\overline{F}}) > 0$. Recall that by the construction, we have $\mathrm{vol}(-mK_X|_F) \leq v_2$. By the boundedness of $\overline{\mathcal{P}}$ and [**1**, Lemma 2.40] (see Lemma 4.9 of Chapter 2), there is a number p (dependent only on v_2) such that $\mathrm{vol}(pK_{\overline{F}} + g_* A_{F'}) > 2^k v_2$ which is equivalent to $\mathrm{vol}(pK_{F'} + A_{F'}) > 2^k v_2$ since $\overline{F}$ is smooth. On the other hand,

$$\begin{aligned}
\mathrm{vol}\left(\frac{m}{n} K_{F'} + A_{F'}\right) &\leq \mathrm{vol}\left(\frac{m}{n} K_F - mK_X|_F\right) \\
&\leq \mathrm{vol}\left(\frac{m}{n} (K_F + \Delta_F) - mK_X|_F\right) \\
&= \mathrm{vol}\left(\frac{m}{n} (-nK_X|_F) - mK_X|_F\right) \\
&= \mathrm{vol}(-2mK_X|_F) \leq 2^k v_2.
\end{aligned}$$

Hence $\frac{m}{n} \leq p$ in this case and we complete Step 1.

Hence we may assume that $\kappa_\sigma(K_{\overline{F}}) = 0$ from now on.

Step 1.4 Reduce to Proposition 4.1.

Since $\kappa_\sigma(K_{\overline{F}}) = 0$, by [**1**, Lemma 2.37] (see Lemma 4.7 of Chapter 2), there is $r \in \mathbb{N}$ depending only on $\overline{\mathcal{P}}$ such that $h^0(rK_{\overline{F}}) \neq 0$. Then $h^0(rK_F) \neq 0$ and $rK_F \sim T_F$ for some integral divisor $T_F \geq 0$.

Suppose that $T_F \neq 0$, then $(F, (1+r)\Delta_F + T_F)$ is not klt. Since $K_F + (1 + r)\Delta_F + T_F \sim_{\mathbb{Q}} (1+r)(K_F + \Delta_F) \sim_{\mathbb{Q}} -(1+r)nK_X|_F$ is nef, $(\overline{F}, \Lambda_{\overline{F}} + \Delta_{\overline{F}} - \Lambda_{\overline{F}} + g_* f^*(r\Delta_F + T_F))$ is not sub-klt by Exercise 6.4. Recall that $(\overline{F}, \Lambda_{\overline{F}})$ is a sub-τ-lc (which is also sub-$\frac{1}{2}$-lc) sub-pair with $\mathrm{Supp}(\Lambda_{\overline{F}}^{\geq 0}) \subset \Sigma_{\overline{F}}$. Hence we may apply Proposition 2.1 to $L = \Delta_{\overline{F}} - \Lambda_{\overline{F}} + g_* f^*(r\Delta_F + T_F)$ and $\tilde{L} = \frac{n+1+rn}{m} M_{\overline{F}} \sim_{\mathbb{Q}} L$, there is $\lambda \in \mathbb{R}_{>0}$ depending only on $\overline{\mathcal{P}}$ such that $\frac{u(n+1+rn)}{m} > \lambda$, which implies that $\frac{m}{n} < \frac{u(2+r)}{\lambda}$. Hence in this case we complete Step 1.

Suppose that $T_F = 0$, then

$$h^0(-rK_X|_F) = h^0(-r(K_F + \Lambda_F)) = h^0(-r\Lambda_F) > 0,$$

since $\Lambda_F \leq 0$ by Step 1.2. By Step 1.2 and [**1**, Proposition 3.15] (see Section 3 of Chapter 9), after replacing r with a multiple depending only on $\overline{\mathcal{P}}$, $h^0(-rnK_X) \neq 0$. Hence $nK_X + N \sim_{\mathbb{Q}} 0$ for some $\mathbb{Q}$-divisor N with coefficients at least $\frac{1}{r}$. Now we can apply Proposition 4.1 to show that $\frac{m}{n} < v$ for a number v depending only on d.

Step 2. Similar to Proposition 4.2, we show that m is bounded from above by a number depending only on d.

By Step 1, $m/n < v$. We may assume that $n > 1$, otherwise $m < v$ and there is nothing to prove. Now the assumptions of Lemma 3.3 are satisfied. Hence there exists a log bounded family $\overline{\mathcal{P}}'$ of couples such that there exists a couple $(\overline{W}, \Sigma_{\overline{W}}) \in \overline{\mathcal{P}}'$ satisfying the following:

- X is birational to $\overline{W}$, $(\overline{W}, \Sigma_{\overline{W}})$ is log smooth, where we may take a higher model of W such that the induced map $\psi : W \dashrightarrow \overline{W}$ is a morphism, where $\phi : W \to X$ satisfies the assumptions in Notation 3.2;
- $\Sigma_{\overline{W}}$ consists of the support of the strict transform of Δ_m and divisors exceptional over X where Δ_m is defined in Notation 3.2;
- the coefficients of $\psi_*\phi^*\Delta_m$ are bounded from above by a number, say u', depending only on d, ϵ, and v.

Note that $K_{\overline{W}}$ is not pseudo-effective as X is Fano, by the boundedness of $\overline{\mathcal{P}}'$ and [**1**, Lemma 2.35] (see Lemma 4.8 of Chapter 2), there exists a number λ'' depending only on $\overline{\mathcal{P}}'$ such that $K_{\overline{W}} + \lambda''\Sigma_{\overline{W}}$ is not pseudo-effective. Write $K_{\overline{W}} + \Lambda_{\overline{W}} := \psi_*\phi^*K_X$. Note that $\mathrm{Supp}(\Lambda_{\overline{W}}) \subset \Sigma_{\overline{W}}$ by the construction since $\Lambda_{\overline{W}}$ is exceptional over X. Since X is τ-lc, the coefficients of $\Lambda_{\overline{W}}$ are at most $1-\tau$. By the construction,

$$K_{\overline{W}} + \Lambda_{\overline{W}} + \frac{1}{m}\psi_*\phi^*\Delta_m \sim_{\mathbb{Q}} \psi_*\phi^*\Big(K_X + \frac{1}{m}\Delta_m\Big) \sim_{\mathbb{Q}} 0.$$

which implies that $K_{\overline{W}} + \Lambda^{\geq 0}_{\overline{W}} + \frac{1}{m}\psi_*\phi^*\Delta_m$ is pseudo-effective. Note that $\Lambda^{\geq 0}_{\overline{W}} + \frac{1}{m}\psi_*\phi^*\Delta_m$ is supported on $\Sigma_{\overline{W}}$ with coefficients at most $1-\tau+\frac{u'}{m}$, hence $1-\tau+\frac{u'}{m} > \lambda''$. We may assume that $\tau > 1-\frac{\lambda''}{2}$, which implies that $m < \frac{2u'}{\lambda''}$ and we are done.

Finally, we summarize the restrictions on τ. We need $\tau > 1-\frac{1}{4}\min\Phi^{>0}$ in Step 1.1, $\tau > 1-\frac{\lambda'}{2}$ in Step 1.3, $\tau > 1-\frac{\lambda''}{2}$ in Step 2. All these restrictions depend only on d. □

6. Exercises

Exercise 6.1. *Let X be a projective variety of dimension d, H a very ample divisor, and L an effective $\mathbb{R}$-divisor. Then* $\mathrm{mult}_x L \leq L\cdot H^{d-1}$ *for any point $x \in X$.*

Exercise 6.2. *Let X be a smooth variety, B an effective $\mathbb{R}$-divisor with simple normal crossing support, and L an effective $\mathbb{R}$-divisor. Fix $\epsilon > 0$. Suppose that (X, B) is ϵ-lc (or equivalently, the coefficients of B are at most $1-\epsilon$) and* $\mathrm{mult}_x L < \epsilon$ *for any point $x \in X$. Then $(X, B+L)$ is klt.*

Exercise 6.3. *Let $\phi : W \dashrightarrow Y$ be a birational contraction between normal projective varieties. Let D be an $\mathbb{R}$-divisor on W. Show that* $\mathrm{vol}(D) \leq \mathrm{vol}(\phi_*D)$. *In particular, if D is big, so is ϕ_*D.*

Exercise 6.4. *Let X and Y be two birational equivalent normal projective varieties. Take a common resolution $\phi : W \to X$ and $\psi : W \to Y$. Assume that (X, Δ) is a sub-pair such that (X, Δ) is not sub-klt and $K_X + \Delta$ is nef. Assume that $K_Y + \Delta_Y := \psi_*\phi^*(K_X + \Delta)$ is $\mathbb{R}$-Cartier. Show that (Y, Δ_Y) is not sub-klt.*

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